

## NOTES ON GEMETRIC LANGLANDS: DG INDSCHMES

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ABSTRACT. We develop the notion of indscheme in the context of derived algebraic geometry, and study the categories of quasi-coherent sheaves and ind-coherent sheaves on indschemes. The main results concern the relation between classical and derived indschemes and the notion of formal smoothness.

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## INTRODUCTION

**0.1. What is this paper about?** The goal of this paper is to develop the foundations of the theory of indschemes, especially in the context of derived algebraic geometry.

0.1.1. The first question to ask here is "why bother"? For, it is more or less clear what DG indschemes are: functors on the category of affine DG schemes, i.e.,  $\infty$ -prestacks in the terminology of [GL:Stacks], that can be written as filtered colimits of DG schemes with transition maps being closed embeddings.

The definition of the category of quasi-coherent sheaves on a DG indscheme  $\mathcal{X}$  is also automatic: the category  $\mathrm{QCoh}(\mathcal{X})$  is defined on any  $\infty$ -prestack (see [GL:QCoh], Sect.1.1 or [Lu1], Sect. 2.7), and in particular on a DG indscheme.

Here is, however, the question, which started life as a remark in another paper, but answering which in detail was one of the main reasons for writing the present one:

0.1.2. Consider the affine Grassmannian  $\mathrm{Gr}_G$  corresponding to an algebraic group  $G$ . This is an indscheme that figures prominently in the geometric Langlands theory. We would like to consider the category  $\mathrm{QCoh}(\mathrm{Gr}_G)$  of quasi-coherent sheaves on  $\mathrm{Gr}_G$ .<sup>1</sup> However, a moment's reflection leads one to conclude that the expression  $\mathrm{QCoh}(\mathrm{Gr}_G)$  is ambiguous. Namely, the affine Grassmannian itself can be understood in two, a priori different, ways.

Recall that, as a functor on the category of commutative algebras,  $\mathrm{Gr}_G$  assigns to an algebra  $A$  the groupoid of  $G$ -torsors over  $\mathrm{Spec}(A[[t]])$  with a trivialization over  $\mathrm{Spec}(A((t)))$ .

Now, we can first take  $A$ 's to be *classical*, i.e., non-derived, commutative algebras, and thus consider  $\mathrm{Gr}_G$  as a classical indscheme. Let us denote this version of  $\mathrm{Gr}_G$  by  ${}^{cl}\mathrm{Gr}_G$ . As for any classical indscheme, we can consider the category  $\mathrm{QCoh}({}^{cl}\mathrm{Gr}_G)$ .

The second possibility is to take  $A$ 's to be DG algebras, and thus consider  $\mathrm{Gr}_G$  right away as an object of derived algebraic geometry. Thus, we obtain a different version of  $\mathrm{QCoh}(\mathrm{Gr}_G)$ .

There is a natural functor

$$(0.1) \quad \mathrm{QCoh}(\mathrm{Gr}_G) \rightarrow \mathrm{QCoh}({}^{cl}\mathrm{Gr}_G),$$

and our initial question was whether or not it is an equivalence.

If it were not an equivalence, it would signify substantial trouble for the geometric Langlands community: on the one hand,  ${}^{cl}\mathrm{Gr}_G$  is a familiar object that people have dealt with for some time now. However, it is clear that the  $\mathrm{Gr}_G$  is "the right object to consider" if we ever want to mix derived algebraic geometry into our considerations, which we inevitably do.<sup>2</sup>

To calm the anxious reader, let us say that the functor (0.1) is an equivalence, as is guaranteed by Theorem 9.3.4 of the present paper.

In fact, we show that  $\mathrm{Gr}_G$  is "the same as"  ${}^{cl}\mathrm{Gr}_G$ , in the sense that the former is obtained from the latter by the natural procedure of turning classical schemes/indschemes/ $\infty$ -stacks into

<sup>1</sup>The other main result of this paper, also of direct relevance to geometric Langlands, is described in Sect. 0.3.1 below. It expresses the category  $\mathrm{QCoh}(\mathrm{Gr}_G)$  in terms of the corresponding category of ind-coherent sheaves on  $\mathrm{Gr}_G$ .

<sup>2</sup>One might raise an objection to the relevance of the above question by remarking that for geometric Langlands we mainly consider D-modules on  $\mathrm{Gr}_G$ , and those only depend on the underlying classical indscheme. However, this is not accurate, since along with D-modules, we consider their global sections as quasi-coherent sheaves, and the latter do depend on the scheme-theoretic structure.

derived ones,<sup>3</sup> which preserves the operation of taking QCoh (see Lemma 1.2.5 of [GL:QCoh] for the latter statement).

0.1.3. While we are at it, let us mention another result along these lines, Proposition 6.7.2.

Let  $X$  be a classical scheme and  $Y \subset X$  a Zariski-closed subset. Consider the formal completion  $X_Y^\wedge$ . By definition, as a functor on commutative algebras,  $X_Y^\wedge$  assigns to a ring  $A$  the groupoid of maps  $\mathrm{Spec}(A) \rightarrow X$ , such that their image is, set-theoretically, contained in  $Y$ .

However, again there are two ways to understand  $X_Y^\wedge$ : as a classical indscheme, which we then turn into a DG indscheme by the procedure mentioned above. Or, we can consider it as a functor of DG algebras, obtaining a DG indscheme right away.

In Proposition 6.7.2 we show that, under the assumption that  $X$  is Noetherian, the above two versions of  $X_Y^\wedge$  are isomorphic.

So, by and large, this paper is devoted to developing the theory to prove the above and similar results.

**0.2. What is done in this paper.** We shall presently proceed to review the main results of this paper (not necessarily in the order in which they appear in the paper).

We should say that none of these results is really surprising. Rather, they are all in the spirit of “things work as they should.”<sup>4</sup>

**0.2.1. DG indschemes via deformation theory.** The first theorem of this paper, Theorem 5.1.1, addresses the following issue. Let  $\mathcal{X}$  be an  $\infty$ -prestack, such that the underlying classical  $\infty$ -prestack is a classical indscheme. What are the conditions that would guarantee that  $\mathcal{X}$  is itself a DG indscheme?

There is an obvious guess: since DG algebras can be thought of as infinitesimal deformations of classical algebras, if we know the behavior of the functor  $\mathcal{X}$  on the latter, its behavior on the former should be governed by deformation theory.

By deformation theory we mean the usual thing: if an algebra  $A'$  is the extension of an algebra  $A$  by a square-zero ideal  $\mathcal{I}$ , then the groupoid of extensions of a given map  $x : \mathrm{Spec}(A) \rightarrow \mathcal{X}$  to a map  $x' : \mathrm{Spec}(A') \rightarrow \mathcal{X}$  is determined by the *cotangent space* to  $\mathcal{X}$  at  $x$ , denoted  $T_x^*\mathcal{X}$ , which is understood just as a functor on the category of  $\mathcal{I}$ 's, i.e., on  $A$ -mod.

If we expect  $\mathcal{X}$  to be a DG indscheme, then the functor

$$(0.2) \quad T_x^*\mathcal{X} : A\text{-mod} \rightarrow \infty\text{-Grpd}$$

must have certain properties: first for a given algebra  $A$ , and also for how it behaves under algebra homomorphisms  $A \rightarrow B$ . If an abstract  $\infty$ -prestack  $\mathcal{X}$  has these properties, we shall say that  $\mathcal{X}$  *admits connective deformation theory*.

Our Theorem 5.1.1 asserts that if  $\mathcal{X}$  is such that its underlying classical  $\infty$ -prestack is a classical indscheme, and if  $\mathcal{X}$  admits connective deformation theory, then it is a DG indscheme.

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<sup>3</sup>This procedure is the left Kan extension along the embedding  $\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ , followed by sheafification in the fppf topology.

<sup>4</sup>For the duration of the paper we make the technical assumption that our DG indscheme are what one could call “ind-quasi-compact and “ind-quasi-separated.”

0.2.2. *Formal smoothness.* Let us recall the notion of formal smoothness for a classical scheme, or more generally for a classical  $\infty$ -prestack, i.e. a functor

$$(0.3) \quad \mathcal{X} : (\mathrm{Sch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}.$$

We say that  $\mathcal{X}$  is *formally smooth* if whenever  $S \rightarrow S'$  is a nil-immersion (i.e., a closed embedding that induces an isomorphism of the underlying reduced classical schemes), then the restriction map

$$\pi_0(\mathcal{X}(S')) \rightarrow \pi_0(\mathcal{X}(S))$$

is surjective.

The notion of formal smoothness in the DG setting is less evident. We formulate it as follows. Let  $\mathcal{X}$  be an  $\infty$ -prestack, i.e., just a functor

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}.$$

We say that it is formally smooth if:

- When we restrict  $\mathcal{X}$  to classical affine schemes, the resulting functor as in (0.3) be formally smooth in the classical sense.
- For an affine DG scheme  $S = \mathrm{Spec}(A)$ , the  $i$ -th homotopy group of the  $\infty$ -groupoid  $\mathcal{X}(S)$  depends only on the truncation  $\tau^{\geq -i}(A)$  (i.e., a map  $A_1 \rightarrow A_2$  that induces an isomorphism of the  $i$ -th truncations should induce an isomorphism of  $\pi_i$ 's of  $\mathcal{X}(\mathrm{Spec}(A_1))$  and  $\mathcal{X}(\mathrm{Spec}(A_2))$ ).<sup>5</sup>

It is well-known that if a classical scheme *of finite type* is classically formally smooth, then it is actually smooth. This implies that it is formally smooth also when viewed as a derived scheme.<sup>6</sup>

The question we would like to ask is whether the same is true for ind schemes. Namely, if  $\mathcal{X}$  is a classical indscheme, which is classically formally smooth, and *locally of finite type*, is it true that it will be formally smooth also as a DG indscheme? (By “as a DG indscheme” we mean the procedure of turning classical  $\infty$ -stacks into derived ones by the procedure mentioned above.)

The answer turns out to be “yes”, under some additional technical hypotheses, see Theorem 9.1.2.

Moreover, the above theorem formally implies that (under the same additional hypotheses), every formally smooth DG indscheme is classical, i.e., is obtained by the above procedure from a classical formally smooth indscheme.

The theorem about the affine Grassmannian mentioned above is an easy corollary of this result.

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<sup>5</sup>It is quite possible that a more reasonable definition in both the classical and derived contexts is when the corresponding properties take place not “on the nose”, but after Zariski/Nisnevich/étale localization. It is likely that the notion of formal smoothness defined as above is only sensible for  $\infty$ -prestacks that are “locally of finite type”, or more generally of Tate type.

<sup>6</sup>We do not know whether the latter is true in general without the finite type hypothesis.

0.2.3. *Loop spaces.* We have no idea whether Theorem 9.1.2 remains valid if one omits the locally finite type hypothesis. It is quite possible that this hypothesis is essential. However, we do propose the following conjecture:

Let  $Z$  be a classical affine scheme of finite type, which is smooth. Consider the corresponding DG indscheme  $Z((t))$  (see Sect. 9.2 for the definition). It is easy to see that it is formally smooth.

We conjecture that, although  $Z((t))$  is not locally of finite type, it is classical. The evidence for this is provided by [Dr], Theorem 6.4. This theorem says that  $Z((t))$  violates the locally finite type condition by factors isomorphic to the infinite-dimensional affine space, and the latter does not affect the property of being classical.

We prove this conjecture in the special case when  $Z$  is an algebraic group  $G$ .

0.3. **Quasi-coherent and ind-coherent sheaves on indschemes.** With future applications in mind, the focus of this paper is the categories  $\mathrm{IndCoh}(\mathcal{X})$  and  $\mathrm{QCoh}(\mathcal{X})$  of ind-coherent and quasi-coherent on a DG indscheme  $\mathcal{X}$ .<sup>7</sup>

We shall now proceed to state the main result of this paper.

0.3.1. *Comparison  $\mathrm{QCoh}$  and  $\mathrm{IndCoh}$  on the loop group.* Let us return to the situation of the affine Grassmannian  $\mathrm{Gr}_G$ , or rather, the loop group  $G((t))$ . As we now know, both of these DG indschemes are classical.

In the study of local geometric Langlands, one considers the notion of a category acted on by the loop group  $G((t))$ . This notion may be defined in two, a priori, different ways:

- (a) As a *co-action* of the *co-monoidal* category  $\mathrm{QCoh}(G((t)))$ , where the co-monoidal structure is given by pullback with respect to the multiplication map on  $G((t))$ .
- (b) As an *action* of the *monoidal* category  $\mathrm{IndCoh}(G((t)))$ , where the monoidal structure is given by push-forward with respect to the same multiplication map.<sup>8</sup>

Obviously, one would like these two notions to coincide. This leads one to believe that the corresponding categories  $\mathrm{QCoh}(G((t)))$  and  $\mathrm{IndCoh}(G((t)))$  are duals of one another (duality is understood here in the sense of [GL:DG], Sect. 2.1).

Moreover, unless we prove something about  $\mathrm{QCoh}(G((t)))$ , it would be a rather unwieldy object, as  $\mathrm{QCoh}(\mathcal{X})$  is for a general DG indscheme  $\mathcal{X}$ . For instance, we would not know that it is compactly generated, etc.

0.3.2. To formulate a precise statement, we shall return to the case of the affine Grassmannian. We claim that the functor

$$\mathrm{QCoh}(\mathrm{Gr}_G) \rightarrow \mathrm{IndCoh}(\mathrm{Gr}_G)$$

given by tensoring with the dualizing sheaf  $\omega_{\mathrm{Gr}_G} \in \mathrm{IndCoh}(\mathrm{Gr}_G)$  is an equivalence.

In fact, we prove Theorem 10.0.6 that asserts that a similarly defined functor is an equivalence for any formally smooth DG indscheme locally of finite type (with an additional technical hypothesis).

This theorem was originally stated and proved by J. Lurie in 2008.

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<sup>7</sup>We refer the reader to [GL:IndCoh] where the category  $\mathrm{IndCoh}(\mathcal{X})$  on a prestack  $\mathcal{X}$  is studied. For it to be defined,  $\mathcal{X}$  needs to be locally almost of finite type, see [GL:Stacks], Sect. 1.3.9.

<sup>8</sup>We should remark that when talking about  $\mathrm{IndCoh}(G((t)))$ , we are leaving the realm of documented mathematics, as  $G((t))$  is not locally of finite type. However, it is not difficult to give a definition of  $\mathrm{IndCoh}$  “by hand” in the particular case of  $G((t))$ , using the affine Grassmannian.

We give a different proof, but it should be noted that Lurie’s original proof was much more elegant. The reason we do not reproduce it here is that it uses some not yet documented facts about Ext computations on indschemes.

**0.3.3. QCoh and IndCoh on formal completions.** Let us mention one more set of results concerning QCoh and IndCoh that we establish in this paper.

In order to prove Theorem 10.0.6 mentioned above, we have to analyze in detail the behavior of the categories QCoh and IndCoh on a DG indscheme obtained as a formal completion  $X_Y^\wedge$  of a DG scheme  $X$  along a Zariski-closed subset  $Y$ .

We show that the category  $\mathrm{QCoh}(X_Y^\wedge)$  (resp.,  $\mathrm{IndCoh}(X_Y^\wedge)$ ) is equivalent to the localization of  $\mathrm{QCoh}(X)$  (resp.,  $\mathrm{IndCoh}(X)$ ) with respect to  $\mathrm{QCoh}(U)$  (resp.,  $\mathrm{IndCoh}(U)$ ), where  $U = X - Y$ .

This implies some favorable properties of  $\mathrm{QCoh}(X_Y^\wedge)$ , e.g., that it is compactly generated (something, which is not necessarily true for an arbitrary indscheme). We also endow  $\mathrm{QCoh}(X_Y^\wedge)$  with two different t-structures, one compatible with pullbacks from  $X$ , and another compatible with push-forwards to  $X$ .

In addition, we show that the functors  $\Psi, \Xi, \Psi^\vee, \Xi^\vee$  that act between QCoh and IndCoh (see [GL:IndCoh], Sect. 1.4 and 8.5) are compatible for  $X_Y^\wedge$  and  $X$  under the push-forward and pullback functors.

**0.4. Conventions and notation.** Our conventions follow closely those of [GL:IndCoh]. Let us recall the most essential ones.

**0.4.1. The ground field.** Throughout the paper we will be working over a fixed ground field  $k$ . To be consistent with [GL:IndCoh], we assume that  $\mathrm{char}(k) = 0$ . However, none of the results of this paper depend on this assumption.

**0.4.2.  $\infty$ -categories.** By an  $\infty$ -category we shall always mean an  $(\infty, 1)$ -category. By a slight abuse of language we will sometimes talk about “categories” when we actually mean  $\infty$ -categories. Our usage of  $\infty$ -categories is not tied to any particular model, but it is their realization as quasi-categories that we actually have in mind, the basic reference to which is [Lu0].

By  $\infty\text{-Grpd}$  we denote the  $\infty$ -category of  $\infty$ -groupoids, which is the same as the category  $\mathcal{S}$  of spaces in the notation of [Lu0].

For an  $\infty$ -category  $\mathbf{C}$ , and  $x, y \in \mathbf{C}$ , we shall denote by  $\mathrm{Maps}_{\mathbf{C}}(x, y) \in \infty\text{-Grpd}$  the corresponding mapping space. By  $\mathrm{Hom}_{\mathbf{C}}(x, y)$  we denote the set  $\pi_0(\mathrm{Maps}_{\mathbf{C}}(x, y))$ , i.e., what is denoted  $\mathrm{Hom}_{h\mathbf{C}}(x, y)$  in [Lu0].

When working in a fixed  $\infty$ -category  $\mathbf{C}$ , for two objects  $x, y \in \mathbf{C}$ , we shall call a point of  $\mathrm{Maps}_{\mathbf{C}}(x, y)$  an *isomorphism* what is in [Lu0] is called an *equivalence*. I.e., a map that admits a homotopy inverse. We reserve the word “equivalence” to mean a (homotopy) equivalence between  $\infty$ -categories.

**0.4.3. DG categories.** Our conventions regarding DG categories follow [GL:IndCoh], Sect. 0.6.4. Unless specified otherwise, we will only consider continuous functors between DG categories (i.e., exact functors that commute with direct sums, or equivalently, with all colimits). In other words, we will be working in the category  $\mathrm{DGCat}_{\mathrm{cont}}$  in the notation of [GL:DG].<sup>9</sup>

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<sup>9</sup>One can replace  $\mathrm{DGCat}_{\mathrm{cont}}$  by (the equivalent)  $(\infty, 1)$ -category of stable presentable  $\infty$ -categories tensored over  $\mathrm{Vect}$ , with colimit-preserving functors.

For a DG category  $\mathbf{C}$  equipped with a t-structure, we denote by  $\mathbf{C}^{\leq n}$  (resp.,  $\mathbf{C}^{\geq m}$ ,  $\mathbf{C}^{\leq n, \geq m}$ ) the corresponding full subcategory of  $\mathbf{C}$  spanned by objects  $x$ , such that  $H^i(x) = 0$  for  $i > n$  (resp.,  $i < m$ ,  $(i > n) \wedge (i < m)$ ). The inclusion  $\mathbf{C}^{\leq n} \hookrightarrow \mathbf{C}$  admits a right adjoint denoted by  $\tau^{\leq n}$ , and similarly, for the other categories.

0.4.4. *(Pre)stacks and DG schemes.* Our conventions regarding (pre)stacks and DG schemes follow [GL:Stacks]:

Let  $\mathrm{DGSch}^{\mathrm{aff}}$  denote the  $\infty$ -category opposite to that of *connective* commutative DG algebras over  $k$ .

The category  $\mathrm{PreStk}$  of prestacks is by definition that of all functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}.$$

The category  $\mathrm{Stk}$  is a full subcategory in  $\mathrm{PreStk}$  that consists of those functors that satisfy fpppf descent (see [GL:Stacks], Sect. 2.2.). This inclusion admits a left adjoint, denoted  $L$ , referred to as the *sheafification functor*.

0.5. **The notion of  $n$ -coconnectivity for (pre)stacks.** For the reader's convenience, in this subsection, we briefly review the material of [GL:Stacks] related to the notion of  $n$ -connectivity.

0.5.1. Let  $n$  be a non-negative integer.

We denote by  $^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$  the full subcategory of  $\mathrm{DGSch}^{\mathrm{aff}}$  that consists of affine DG schemes  $S = \mathrm{Spec}(A)$ , such that  $H^{-i}(A) = 0$  for  $i > n$ . We shall refer to objects of this category as “ $n$ -coconnective affine DG schemes”. When  $n = 0$  we shall also use the terminology “classical affine schemes”, and denote this category by  $\mathrm{Sch}^{\mathrm{aff}}$ .

The inclusion  $^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$  admits a right adjoint given by cohomological truncation above degree  $-n$ ; we denote this functor by  $S \mapsto \tau^{\leq n}(S)$ .

0.5.2. *The case of prestacks.* In this paper, we make extensive use of the operation of restricting a prestack  $\mathcal{Y}$  to the subcategory  $^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ . We denote this functor by

$$\mathcal{Y} \mapsto ^{\leq n}\mathcal{Y} : \mathrm{PreStk} \rightarrow ^{\leq n}\mathrm{PreStk},$$

where  $^{\leq n}\mathrm{PreStk}$  is by definition the category of all functors  $(^{\leq n}\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}$ .

The above restriction functor admits a (fully faithful) left adjoint, given by left Kan extension along  $^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}$ ; we denote it by

$$\mathrm{LKE}_{^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : ^{\leq n}\mathrm{PreStk} \rightarrow \mathrm{PreStk}.$$

The composition

$$\mathcal{Y} \mapsto \mathrm{LKE}_{^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} (^{\leq n}\mathcal{Y})$$

is a colocalization functor on  $\mathrm{PreStk}$ ; we denote it by  $\mathcal{Y} \mapsto \tau^{\leq n}(\mathcal{Y})$ . When  $\mathcal{Y}$  is an affine scheme  $S$ , this coincides with what was denoted above by  $\tau^{\leq n}(S)$ .

We shall say that a prestack  $\mathcal{Y}$  is  *$n$ -coconnective* if it belongs to the essential image of  $\mathrm{LKE}_{^{\leq n}\mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}$ , or equivalently if the canonical map  $\tau^{\leq n}(\mathcal{Y}) \rightarrow \mathcal{Y}$  is an isomorphism.

Thus, the functors of restriction and left Kan extension identify  $^{\leq n}\mathrm{PreStk}$  with the full subcategory of  $\mathrm{PreStk}$  spanned by  *$n$ -coconnective* prestacks.

We shall say that  $\mathcal{Y}$  is *eventually coconnective* if it is  $n$ -coconnective for some  $n$ .

We shall refer to objects of  $^{\leq 0}\mathrm{PreStk}$  as “classical prestacks”; we shall denote this category also by  $^{\mathrm{cl}}\mathrm{PreStk}$ . By the above, the category of classical prestacks is canonically equivalent to that of 0-coconnective prestacks.



0.5.3. *The notion of  $n$ -coconnectivity for stacks.* By considering fppf topology on the category  $\leq^n \text{DGSch}^{\text{aff}}$ , we obtain the corresponding full subcategory

$$\leq^n \text{Stk} \subset \leq^n \text{PreStk}.$$

The restriction functor  $\text{PreStk} \rightarrow \leq^n \text{PreStk}$  sends

$$(0.4) \quad \text{Stk} \rightarrow \leq^n \text{Stk},$$

but the left adjoint  $\text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$  does not send  $\leq^n \text{Stk}$  to  $\text{Stk}$ . The left adjoint to the functor (0.4) is given by the composition

$$\leq^n \text{Stk} \hookrightarrow \leq^n \text{PreStk} \xrightarrow{\text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}} \text{PreStk} \xrightarrow{L} \text{Stk},$$

and is denoted  ${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$ . The functor  ${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$  is fully faithful. The composition of the functor (0.4) with  ${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$  is a colocalization functor on  $\text{Stk}$  and is denoted  $\mathcal{Y} \mapsto {}^L \tau^{\leq n}(\mathcal{Y})$ .

We shall say that a stack  $\mathcal{Y} \in \text{Stk}$  is  *$n$ -coconnective as a stack* if it belongs to the essential image of the functor  ${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$ , or equivalently, if the canonical map  ${}^L \tau^{\leq n}(\mathcal{Y}) \rightarrow \mathcal{Y}$  is an isomorphism.

We emphasize, however, that if  $\mathcal{Y}$  is  *$n$ -coconnective as a stack*, it is *not* necessarily  *$n$ -coconnective as a prestack*. The corresponding morphism  $\tau^{\leq n}(\mathcal{Y}) \rightarrow \mathcal{Y}$  becomes an isomorphism only after applying the sheafification functor  $L$ .

Thus, the functor (0.4) and its left adjoint identify the category  $\leq^n \text{Stk}$  with the full subcategory of  $\text{Stk}$  spanned by  *$n$ -coconnective stacks*.

We shall say that  $\mathcal{Y}$  is *eventually coconnective as a stack* if it is  *$n$ -coconnective as a stack* for some  $n$ .

We shall refer to objects of  $\leq^0 \text{Stk}$  as “classical stacks”; we shall also denote this category by  ${}^c \text{Stk}$ . By the above, the category of classical stacks is canonically equivalent to that of 0-coconnective stacks.

0.5.4. *DG schemes.* The category  $\text{Stk}$  (resp.,  $\leq^n \text{Stk}$ ) contains the full subcategory  $\text{DGSch}$  (resp.,  $\leq^n \text{DGSch}$ ), see [GL:Stacks], Sect. 3.2.

The functors of restriction and  ${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$  send the categories  $\text{DGSch}$  and  $\leq^n \text{DGSch}$  to one another, thereby identifying  $\leq^n \text{DGSch}$  with the subcategory of  $\text{DGSch}$  that consists of  *$n$ -coconnective DG schemes*, i.e., those DG schemes that are  *$n$ -coconnective as stacks*.

In order to avoid unbearably long formulas, we will sometimes use the following slightly abusive notation: if  $Z$  is an object of  $\leq^n \text{DGSch}$ , we will use the same symbol  $Z$  for the object of  $\text{DGSch}$  that should properly be denoted

$${}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(Z).$$

Similarly, for  $n' \geq n$ , we shall write  $Z$  for the object of  $\leq^{n'} \text{DGSch}$  that should properly be denoted

$$\leq^{n'} {}^L \text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \leq^{n'} \text{DGSch}^{\text{aff}}}(Z_n).$$

For  $n = 0$  we shall refer to objects of  $\leq^0 \text{DGSch}$  as “classical schemes”, and denote this category also by  $\text{Sch}$ .

0.5.5. *Convergence.* An object  $\mathcal{Y}$  of  $\text{PreStk}$  (resp.,  $\text{Stk}$ ) is said to be convergent if for any  $S \in \text{DGSch}^{\text{aff}}$ , the natural map

$$\mathcal{Y}(S) \rightarrow \lim_n \mathcal{Y}(\tau^{\leq n}(S))$$

is an isomorphism.

Equivalently,  $\mathcal{Y} \in \text{PreStk}$  (resp.,  $\text{Stk}$ ) is convergent if the map

$$\mathcal{Y} \rightarrow \text{RKE}_{<\infty \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}|_{<\infty \text{DGSch}^{\text{aff}}})$$

is an isomorphism. Here,  $<\infty \text{DGSch}^{\text{aff}}$  denotes the full subcategory of  $\text{DGSch}^{\text{aff}}$  spanned by eventually coconnective affine DG schemes.

The full subcategory of  $\text{PreStk}$  (resp.,  $\text{Stk}$ ) that consists of convergent objects is denoted  ${}^{\text{conv}}\text{PreStk}$  (resp.,  ${}^{\text{conv}}\text{Stk}$ ). The embedding

$${}^{\text{conv}}\text{PreStk} \hookrightarrow \text{PreStk}$$

admits a left adjoint, called the convergent completion, and denoted  $\mathcal{Y} \mapsto {}^{\text{conv}}\mathcal{Y}$ .<sup>10</sup> The restriction of this functor to  $\text{Stk}$  sends

$$\text{Stk} \rightarrow {}^{\text{conv}}\text{Stk},$$

and is the left adjoint to the embedding  ${}^{\text{conv}}\text{Stk} \hookrightarrow \text{Stk}$ .

Tautologically, we can describe the functor of convergent completion as the composition

$$\mathcal{Y} \mapsto \text{RKE}_{<\infty \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}|_{<\infty \text{DGSch}^{\text{aff}}}).$$

I.e., the functor of right Kan extension  $\text{RKE}_{<\infty \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}$  along  $<\infty \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}$  identifies the category  $<\infty \text{PreStk}$  with  ${}^{\text{conv}}\text{PreStk}$ , and  $<\infty \text{Stk}$  with  ${}^{\text{conv}}\text{Stk}$ .

0.5.6. *Weak  $n$ -coconnectivity.* For a fixed  $n$ , the composite functor

$${}^{\text{conv}}\text{PreStk} \hookrightarrow \text{PreStk} \xrightarrow{\text{restriction}} {}^{\leq n}\text{PreStk}$$

also admits a left adjoint given by

$$(0.5) \quad \mathcal{Y}_n \mapsto {}^{\text{conv}}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}_n) := {}^{\text{conv}}(\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}_n)).$$

Equivalently, when we identify  $<\infty \text{PreStk} \simeq {}^{\text{conv}}\text{PreStk}$ , the above functor can be described as  $\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow <\infty \text{DGSch}^{\text{aff}}}$ .

The composite functor

$$\mathcal{Y} \mapsto {}^{\text{conv}}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}|_{\leq n \text{DGSch}^{\text{aff}}})$$

is a colocalization on  ${}^{\text{conv}}\text{PreStk}$ , and we will denote it by  ${}^{\text{conv}}\tau^{\leq n}$ .

Similarly, the composite functor

$${}^{\text{conv}}\text{Stk} \hookrightarrow \text{Stk} \xrightarrow{\text{restriction}} {}^{\leq n}\text{Stk}$$

also admits a left adjoint given by

$$(0.6) \quad \mathcal{Y}_n \mapsto {}^{\text{conv}, L}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}_n) := {}^{\text{conv}}({}^L\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}_n)).$$

Alternatively, when we identify  $<\infty \text{Stk} \simeq {}^{\text{conv}}\text{Stk}$ , the above functor can be described as

$$<\infty {}^L\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow <\infty \text{DGSch}^{\text{aff}}}.$$

The composite functor

$$\mathcal{Y} \mapsto {}^{\text{conv}, L}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}|_{\leq n \text{DGSch}^{\text{aff}}})$$

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<sup>10</sup>In [GL:Stacks], this functor was denoted  $\mathcal{Y} \mapsto \hat{\mathcal{Y}}$ .

is a colocalization on  ${}^{\text{conv}}\text{Stk}$ , and we will denote it by  ${}^{\text{conv}, L_T \leq n}$ .

We shall say that an object  $\mathcal{Y}$  of  ${}^{\text{conv}}\text{PreStk}$  (resp.,  ${}^{\text{conv}}\text{Stk}$ ) is *weakly  $n$ -coconnective* if it belongs to the essential image of the functor (0.5) (resp., (0.6)). Equivalently, an object as above is *weakly  $n$ -coconnective* if and only if its restriction to  ${}^{\leq m}\text{DGSch}^{\text{aff}}$  is  $n$ -coconnective for any  $m \geq n$ .

It is clear that if an object is  $n$ -coconnective, then it is weakly  $n$ -coconnective. However, the converse is false.

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## 1. DG INDSCHEMES

When dealing with usual indschemes, the definition is straightforward: like any "space" in algebraic geometry, an indscheme is a presheaf on the category of affine schemes, and the condition we require is that it should be representable by a filtered family of schemes, where the transition maps are closed embeddings.

The same definition is reasonable in the DG setting as long as we restrict ourselves to  $n$ -coconnective DG schemes for some  $n$ . However, when dealing with arbitrary DG indschemes, one has to additionally require that the presheaf be *convergent*, see Sect. 0.5.5.

Thus, for reasons of technical convenience we define DG indschemes by requiring the existence of a presentation as a filtered colimit *at the truncated level*. We will later show that DG indschemes defined in this way themselves admit a presentation as a colimit of DG schemes.

In this section we define DG indschemes, first in the  $n$ -coconnective setting for some  $n$ , and then in general, and study the relationship between these two notions. We also introduce the categories of quasi-coherent and ind-coherent sheaves on an ind-scheme.

As was mentioned in the introduction, the class of (DG) indschemes that we consider in this paper is somewhat smaller than one could in principle consider in general: we will only consider those (DG) indschemes that are ind-quasi-compact and ind-quasi-separated.

### 1.1. Definition in the $n$ -coconnective case.

1.1.1. Let us recall the notion of closed embedding in derived algebraic geometry.

**Definition 1.1.2.** A map  $X_1 \rightarrow X_2$  in  $\text{DGSch}$  or  ${}^{\leq n}\text{DGSch}$  is a *closed embedding* if the corresponding map of classical schemes  ${}^{cl}X_1 \rightarrow {}^{cl}X_2$  is.

Recall that the notation  ${}^{cl}X$  means  $X|_{{}^{cl}\text{DGSch}^{\text{aff}}}$ , i.e., we regard  $X$  as a functor on classical affine schemes, and if  $X$  was a DG scheme, then  ${}^{cl}X$  is a classical scheme (see [GL:Stacks], Sect. 3.2.1).

Let  $(\text{DGSch})_{\text{closed}}$  (resp.,  $({}^{\leq n}\text{DGSch})_{\text{closed}}$ ) denote the non-full subcategory of  $\text{DGSch}$  (resp.,  ${}^{\leq n}\text{DGSch}$ ), where we restrict 1-morphisms to be closed embeddings.

1.1.3. By a  $\leq^n \text{DG indscheme}$  we shall mean an object  $\mathcal{X}$  of  $\leq^n \text{PreStk}$  which can be represented as a colimit of a functor

$$A \rightarrow \leq^n \text{PreStk}$$

which factors as

$$A \rightarrow (\leq^n \text{DGSch}_{\text{qc-qs}})_{\text{closed}} \hookrightarrow \leq^n \text{PreStk},$$

and where the category  $A$  is filtered. I.e., we need to be able to represent  $\mathcal{X}$  as a filtered colimit in  $\leq^n \text{PreStk}$ :

$$(1.1) \quad \text{colim}_{\alpha} X_{\alpha},$$

where  $X_{\alpha} \in \leq^n \text{DGSch}_{\text{qc-qs}}$  and for  $\alpha_1 \rightarrow \alpha_2$ , the corresponding map  $i_{\alpha_1, \alpha_2} : X_{\alpha_1} \rightarrow X_{\alpha_2}$  is a closed embedding. Here  $\leq^n \text{DGSch}_{\text{qc-qs}}$  is the full subcategory of  $\leq^n \text{DGSch}$  that consists of DG schemes that are quasi-compact and quasi-separated; by definition this is a condition on the underlying classical scheme.

Let  $\leq^n \text{DGindSch}$  denote the full subcategory of  $\leq^n \text{Stk}$  spanned by  $\leq^n \text{DG indscheme}$ s. We shall refer to objects of  $\leq^0 \text{DGindSch}$  as *classical indscheme*s; we shall also use the notation  $\text{indSch} := \leq^0 \text{DGindSch}$ .

*Remark.* Note that the quasi-compactness and quasi-separatedness assumption in the definition of  $\leq^n \text{DG indscheme}$ s means that not every  $\leq^n \text{DG scheme}$   $X$  is a  $\leq^n \text{DG indscheme}$ . However,  $X$  does not have to be quasi-compact and quasi-separated to be a  $\leq^n \text{DG indscheme}$ : for example, a disjoint union of quasi-compact and quasi-separated  $\leq^n \text{DG scheme}$ s is a  $\leq^n \text{DG indscheme}$ .

1.1.4. We observe that a  $\leq^n \text{DG indscheme}$  automatically belongs to  $\leq^n \text{Stk}$ . This follows from the next lemma:

**Lemma 1.1.5.** *Let  $\alpha \mapsto X_{\alpha}$  be a filtered diagram in  $\leq^n \text{PreStk}$ . Set*

$$\mathcal{X} := \text{colim}_{\alpha} X_{\alpha}.$$

*Then if all  $X_{\alpha}$  belong to  $\leq^n \text{Stk}$  and are  $k$ -truncated for some  $k$  (see [GL:Stacks], Sect. 1.1.7), then so does  $\mathcal{X}$ .*

*Proof.* By construction, the functors

$$X_{\alpha} \text{ and } \mathcal{X} : \leq^n \text{DGSch}^{\text{aff}} \rightarrow \infty\text{-Grpd}$$

take values in the subcategory  $(k+n)$ -groupoids. Hence, for an fppf cover  $S' \rightarrow S$  and the corresponding Čech simplicial scheme  $S'_{\bullet}/S$  and its  $(k+n+1)$ -truncation  $S'^{\bullet \leq k+n+1}/S$ , the restriction maps

$$\text{Tot}(X_{\alpha}(S'_{\bullet}/S)) \rightarrow \text{Tot}(X_{\alpha}(S'^{\bullet \leq k+n+1}/S)) \text{ and } \text{Tot}(\mathcal{X}(S'_{\bullet}/S)) \rightarrow \text{Tot}(\mathcal{X}(S'^{\bullet \leq k+n+1}/S))$$

are isomorphisms.

The assertion of the lemma now follows from the fact that finite limits commute with filtered colimits. □

We obtain that if  $\mathcal{X} \in \leq^n \text{Stk}$  is written as in (1.1), *but where the colimit is taken in the category  $\leq^n \text{Stk}$* , then  $\mathcal{X}$  is a  $\leq^n \text{DG indscheme}$ :

Indeed the above lemma implies that the natural map from the colimit of (1.1) taken in  $\leq^n \text{PreStk}$  to that in  $\leq^n \text{Stk}$  is an isomorphism.

1.1.6. Let  $Y$  be an object of  $\leq^n \text{DGSch}$ , and let  $\mathcal{X} \in \leq^n \text{DGindSch}$  be presented as in (1.1). We have a natural map

$$(1.2) \quad \text{colim}_{\alpha} \text{Maps}(Y, X_{\alpha}) \rightarrow \text{Maps}(Y, \mathcal{X}).$$

If  $Y$  is affine, the above map is an isomorphism by definition, since colimits in

$$\leq^n \text{PreStk} = \text{Func}(\leq^n \text{DGSch}^{\text{aff}}, \infty\text{-Grpd})$$

are computed object-wise.

For a general  $Y$ , the map (1.2) need not be an isomorphism. However, we have:

**Lemma 1.1.7.** *If  $Y$  is quasi-compact and quasi-separated, then the map (1.2) is an isomorphism.*

*Proof.* This follows from the fact that  $\mathcal{X}$  belongs to  $\leq^n \text{Stk}$ , and that a quasi-compact and quasi-separated DG scheme can be written as a colimit in  $\leq^n \text{Stk}$  of a *finite* diagram whose terms are in  $\leq^n \text{DGSch}^{\text{aff}}$ , and the fact that filtered colimits in  $\infty\text{-Grpd}$  commute with finite limits.  $\square$

## 1.2. Changing $n$ .

1.2.1. Clearly, for  $n' < n$ , the functor

$$\leq^n \text{PreStk} \rightarrow \leq^{n'} \text{PreStk},$$

corresponding to restriction along

$$\leq^{n'} \text{DGSch}^{\text{aff}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}},$$

sends the subcategory  $\leq^n \text{DGindSch}$  to  $\leq^{n'} \text{DGindSch}$ .

Indeed, if  $\mathcal{X}$  is presented as in (1.1), then  $\leq^{n'} \mathcal{X} := \mathcal{X}|_{\leq^{n'} \text{DGSch}^{\text{aff}}}$  can be presented as

$$\text{colim}_{\alpha} (\leq^{n'} X_{\alpha}).$$

Thus, restriction defines a functor

$$\leq^{n'} \text{DGindSch} \leftarrow \leq^n \text{DGindSch}.$$

1.2.2. Vice versa, consider the functor

$$(1.3) \quad \leq^n L \text{LKE}_{\leq^{n'} \text{DGSch}^{\text{aff}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}}} := \leq^n L \circ \text{LKE}_{\leq^{n'} \text{DGSch}^{\text{aff}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}}} : \leq^{n'} \text{Stk} \rightarrow \leq^n \text{Stk},$$

left adjoint to the restriction functor. In the above formula  $\leq^n L : \leq^n \text{PreStk} \rightarrow \leq^n \text{Stk}$  is the sheafification functor, left adjoint to the embedding  $\leq^n \text{Stk} \hookrightarrow \leq^n \text{PreStk}$ .

We claim that it sends  $\leq^{n'} \text{DGindSch}$  to  $\leq^n \text{DGindSch}$ . Indeed, if  $\mathcal{X}' \in \leq^{n'} \text{DGindSch}$  is written as

$$\mathcal{X}' \simeq \text{colim}_{\alpha} X'_{\alpha}, \quad X'_{\alpha} \in \leq^{n'} \text{DGSch},$$

then

$$\leq^n L \text{LKE}_{\leq^{n'} \text{DGSch}^{\text{aff}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}}}(\mathcal{X}') \simeq \text{colim}_{\alpha} X_{\alpha},$$

where

$$X_{\alpha} := \leq^n L \text{LKE}_{\leq^{n'} \text{DGSch}^{\text{aff}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}}}(X'_{\alpha}) \in \leq^n \text{DGindSch}.$$

1.2.3. We obtain a pair of adjoint functors

$$(1.4) \quad {}^{\leq n'}\mathrm{DGindSch} \rightleftarrows {}^{\leq n}\mathrm{DGindSch},$$

with the left adjoint being fully faithful.

An object  $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$  belongs to the essential image of the left adjoint in (1.4) if and only if it is  $n'$ -coconnective as an object of  ${}^{\leq n}\mathrm{Stk}$ , i.e., if it belongs to the essential image of the left adjoint (1.3).

Moreover, if  $\mathcal{X} \in {}^{\leq n}\mathrm{DGindSch}$  has this property, it admits a presentation as in (1.1), where the  $X_\alpha$  are  $n'$ -coconnective.

### 1.3. General DG indschemes.

1.3.1. *Definition.* We say that an object  $\mathcal{X} \in \mathrm{PreStk}$  is a DG indscheme if the following three conditions hold:

- (1)  $\mathcal{X} \in \mathrm{Stk}$
- (2) As an object of  $\mathrm{PreStk}$ ,  $\mathcal{X}$  is convergent (see [GL:Stacks], Sect. 1.2).
- (3) For every  $n$ ,  ${}^{\leq n}\mathcal{X} := \mathcal{X}|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}}$  is a  ${}^{\leq n}\mathrm{DG}$  indscheme.

We shall denote the full subcategory of  $\mathrm{Stk}$  spanned by DG indschemes by  $\mathrm{DGindSch}$ .

We shall prove the following (see also Proposition 1.4.3 below for a more precise assertion):

**Proposition 1.3.2.** *Any DG indscheme  $\mathcal{X}$  can be presented as a filtered colimit in  $\mathrm{PreStk}$*

$$(1.5) \quad \mathrm{colim}_\alpha X_\alpha,$$

where  $X_\alpha \in \mathrm{DGSch}_{\mathrm{qc-qs}}$  and for  $\alpha_1 \rightarrow \alpha_2$ , the corresponding map  $i_{\alpha_1, \alpha_2} : X_{\alpha_1} \rightarrow X_{\alpha_2}$  is a closed embedding.

The above proposition allows us to give the following, in a sense, more straightforward, definition of DG indschemes:

**Corollary 1.3.3.** *An object  $\mathcal{X} \in \mathrm{PreStk}$  is a DG indscheme if and only if:*

- It belongs to  $\mathrm{Stk}$ ;
- It is convergent;
- As an object of  $\mathrm{PreStk}$  it admits a presentation as in (1.5).

1.3.4. Note that unlike the case of  ${}^{\leq n}\mathrm{DG}$  indschemes, an object of  $\mathrm{PreStk}$  written as in (1.5) need *not* be a DG indscheme. Indeed, it can fail to belong to  $\mathrm{Stk}$ , since Lemma 1.1.5 is no longer applicable. In addition, it can fail to be convergent.

However, such a colimit gives rise to a DG indscheme via the following lemma:

**Lemma 1.3.5.** *For  $\mathcal{X} \in \mathrm{PreStk}$  given as in (1.5), the object*

$${}^{\mathrm{conv}}(L(\mathcal{X})) \in \mathrm{PreStk}$$

*belongs to  $\mathrm{DGindSch}$ .*

*Proof.* By definition,  ${}^{\mathrm{conv}}(L(\mathcal{X}))$  is convergent. It belongs to  $\mathrm{Stk}$  by [GL:Stacks], Corollary 2.4.10.

Thus, it remains to show that the restriction of  ${}^{\mathrm{conv}}(L(\mathcal{X}))$  to  ${}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$  belongs to the subcategory  ${}^{\leq n}\mathrm{DGindSch}$ . We have:

$${}^{\mathrm{conv}}(L(\mathcal{X}))|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}} \simeq L(\mathcal{X})|_{{}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}},$$

and

$$L(\mathcal{X})|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}} \simeq {}^{\leq^n} L(\mathcal{X}|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}}),$$

by [GL:Stacks], Corollary 2.4.6.

Now, since the functor of restriction  $\mathrm{PreStk} \rightarrow {}^{\leq^n} \mathrm{PreStk}$  commutes with colimits, we obtain that  $\mathcal{X}|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}} \in {}^{\leq^n} \mathrm{DGindSch}$ , and by Lemma 1.1.5,

$${}^{\leq^n} L(\mathcal{X}|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}}) \simeq \mathcal{X}|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}}.$$

□

1.3.6. As in Sect. 1.1.6 we consider maps into a DG indscheme  $\mathcal{X}$  from an arbitrary DG scheme  $Y$ , and we have the following analog of Lemma 1.1.7 (with the same proof):

**Lemma 1.3.7.** *For  $\mathcal{X} \in \mathrm{Stk}$  written as in (1.5), and  $Y \in \mathrm{DGSch}$ , the natural map*

$$\mathrm{colim}_{\alpha} \mathrm{Maps}(Y, X_{\alpha}) \rightarrow \mathrm{Maps}(Y, \mathcal{X})$$

*is an isomorphism, provided that  $Y$  is quasi-compact and quasi-separated.*

1.4. **The canonical presentation of a DG indscheme.** We shall now formulate a precise version of Proposition 1.3.2, which will be proved in Sect. 3.2 and Sect. 3.5.

1.4.1. Recall that a map  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  in  $\mathrm{PreStk}$  is called a *closed embedding* if the corresponding map  ${}^{cl}\mathcal{Y}_1 \rightarrow {}^{cl}\mathcal{Y}_2$  is a closed embedding (i.e., its base change by an affine scheme yields a closed embedding).

*Remark.* Note that in the DG setting, being a closed embedding does not imply that a map is schematic. Indeed, a closed embedding of a DG scheme into a DG indscheme is typically not schematic.

It is easy to see that for maps  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2 \rightarrow \mathcal{Y}_3$  with the last one is a closed embedding, the map  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  is a closed embedding if and only if  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$  is.

1.4.2. For a DG indscheme  $\mathcal{X}$ , let

$$(\mathrm{DGSch}_{\mathrm{qc-qc}})_{\mathrm{closed in } \mathcal{X}} \subset (\mathrm{DGSch}_{\mathrm{qc-qc}})_{/\mathcal{X}}$$

be the full subcategory, consisting of those objects for which the map  $Z \rightarrow \mathcal{X}$  is a closed embedding in the above sense.

We have:

**Proposition 1.4.3.** *Let  $\mathcal{X}$  be a DG scheme.*

- (a) *The category  $(\mathrm{DGSch}_{\mathrm{qc-qc}})_{\mathrm{closed in } \mathcal{X}}$  is filtered.*
- (b) *The natural map*

$$(1.6) \quad \mathrm{colim}_{Z \in (\mathrm{DGSch}_{\mathrm{qc-qc}})_{\mathrm{closed in } \mathcal{X}}} Z \rightarrow \mathcal{X},$$

*where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism.*

1.4.4. Combined with Lemma 1.3.7, we obtain the following:

**Corollary 1.4.5.** *Let  $\mathcal{X}$  be a DG indscheme. The functor*

$$(1.7) \quad (\mathrm{DGSch}_{\mathrm{qc-qc}})_{\mathrm{closed in } \mathcal{X}} \rightarrow (\mathrm{DGSch}_{\mathrm{qc-qc}})_{/\mathcal{X}}$$

*is cofinal.*

Moreover, this allows us to give the following characterization of DG indscheme among  $\mathrm{Stk}$ :

**Corollary 1.4.6.** *An object  $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{Stk}$  is a DG indscheme if and only if:*

- *The category of closed embeddings  $Z \rightarrow \mathcal{X}$ , where  $Z \in \mathrm{DGSch}_{\mathrm{qc-qc}}$ , is filtered.*
- *The functor (1.7) is cofinal.*

1.4.7. Let us also note that Lemma 1.3.7 implies that for any presentation of a DG indscheme as in Proposition 1.3.2, the tautological map

$$\mathcal{A} \rightarrow (\mathrm{DGSch}_{\mathrm{qc-qc}})_{\mathrm{closed in } \mathcal{X}}$$

is cofinal.

### 1.5. Restricting back to the $n$ -coconnective case.

1.5.1. If  $\mathcal{X}$  is a DG indscheme, then

$$\leq^n \mathcal{X} := \mathcal{X}|_{\leq^n \mathrm{DGSch}^{\mathrm{aff}}}$$

is a  $\leq^n \mathrm{DG}$  indscheme. In particular,  ${}^{\mathrm{cl}}\mathcal{X}$  is a classical indscheme. Thus, we obtain a functor

$$(1.8) \quad \leq^n \mathrm{DGindSch} \leftarrow \mathrm{DGindSch}.$$

1.5.2. Vice versa, if  $\mathcal{X}_n$  is a  $\leq^n \mathrm{DG}$  indscheme, let

$$\mathcal{X} := {}^{\mathrm{conv}, L} \mathrm{LKE}_{\leq^n \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{X}_n)$$

Then by Lemma 1.3.5,  $\mathcal{X}$  is a DG indscheme.

This defines a functor

$$(1.9) \quad \leq^n \mathrm{DGindSch} \rightarrow \mathrm{DGindSch},$$

which is left adjoint to the one in (1.8). It is easy to see that the unit map defines an isomorphism from the identity functor to

$$\leq^n \mathrm{DGindSch} \rightarrow \mathrm{DGindSch} \rightarrow \leq^n \mathrm{DGindSch}.$$

I.e., the functor in (1.9) is fully faithful.

1.5.3. In what follows, we shall say that a DG indscheme is *weakly  $n$ -coconnective* if it is such as an object of  $\mathrm{Stk}$ , see Sect. 0.5.6, i.e., if it belongs to the essential image of the functor (1.9).

Thus, the above functor establishes an equivalence between  $\leq^n \mathrm{DGindSch}$  and the full subcategory of  $\mathrm{DGindSch}$  spanned by *weakly  $n$ -coconnective* DG schemes. In particular, it identifies classical indscheme with weakly 0-coconnective DG indscheme.

We shall say that  $\mathcal{X}$  is *weakly eventually coconnective* if it is weakly  $n$ -coconnective for some  $n$ .

We shall say that a DG indscheme is  *$n$ -coconnective* if it is  $n$ -coconnective as an object of  $\mathrm{Stk}$ , i.e., if it lies in the essential image of the functor

$$(1.10) \quad {}^L \mathrm{LKE}_{\leq^n \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} : \leq^n \mathrm{Stk} \rightarrow \mathrm{Stk}.$$

We shall say that  $\mathcal{X}$  is *eventually coconnective* if it is  $n$ -coconnective for some  $n$ .



1.5.4. *The  $\aleph_0$  condition.* We shall say that  $\mathcal{X} \in {}^{\leq n}\text{DGindSch}$  is  $\aleph_0$  if there exists a presentation as in (1.1) with the category of indices equivalent to the poset  $\mathbb{N}$ .

We shall say that  $\mathcal{X} \in \text{DGindSch}$  is  $\aleph_0$  if for it admits a presentation as in Proposition 1.3.2, with the category of indices equivalent to the poset  $\mathbb{N}$ .

We shall say that  $\mathcal{X} \in \text{DGindSch}$  is *weakly*  $\aleph_0$  if for every  $n$ , the object

$${}^{\leq n}\mathcal{X} \in {}^{\leq n}\text{DGindSch}$$

is  $\aleph_0$ .

## 1.6. The locally almost of finite type condition.

1.6.1. We shall say that  $\mathcal{X} \in {}^{\leq n}\text{DGindSch}$  is locally of finite type if it is such as an object of  ${}^{\leq n}\text{PreStk}$  (see [GL:Stacks], Sect. 1.3.2), i.e., it belongs to  ${}^{\leq n}\text{PreStk}_{\text{lft}}$  in the terminology of *loc.cit.*

We shall denote the full subcategory of  ${}^{\leq n}\text{DGindSch}$  spanned by  ${}^{\leq n}\text{DG}$  indschemes locally of finite type by  ${}^{\leq n}\text{DGindSch}_{\text{lft}}$ .

We shall say that  $\mathcal{X} \in \text{DGindSch}$  is locally almost of finite type if it is such as an object of  $\text{PreStk}$ , see [GL:Stacks], Sect. 1.3.9, i.e., if in the notation of *loc.cit.* it belongs to the subcategory  $\text{PreStk}_{\text{lft}} \subset \text{PreStk}$ . By definition, this means that

$${}^{\leq n}\mathcal{X} \in {}^{\leq n}\text{DGindSch}$$

must be locally of finite type for every  $n$ . We shall denote the full subcategory of  $\text{DGindSch}$  spanned by  $\text{DG}$  indschemes locally almost of finite type by  $\text{DGindSch}_{\text{lft}}$ .

1.6.2. It is natural to wonder whether one can represent objects of  $\text{DGindSch}_{\text{lft}}$  as colimits of objects of  $\text{DGSch}_{\text{aft}}$  under closed embeddings. (We denote by  $\text{DGSch}_{\text{aft}}$  the category of  $\text{DG}$  schemes almost of finite type, i.e.,  $\text{DGSch}_{\text{aft}} := \text{DGSch}_{\text{lft}} \cap \text{DGSch}_{\text{qc}}$ , see [GL:Stacks], 3.3.1.)

In fact, there are two senses in which one can ask this question: one may want to have a presentation in a “weak sense”, i.e., as in Lemma 1.3.5, or in the “strong” sense, i.e., as in Proposition 1.3.2.

The answer to the “weak” version is affirmative: we will prove the following:

**Proposition 1.6.3.** *For a  $\text{DG}$  indscheme  $\mathcal{X}$  locally almost of finite type there exists a filtered family*

$$A \rightarrow (\text{DGSch}_{\text{aft}})_{\text{closed}} : \alpha \mapsto X_\alpha,$$

*such that  $\mathcal{X}$  is isomorphic to the convergent completion of*

$$(1.11) \quad \text{colim}_{\alpha \in A} X_\alpha,$$

*where the colimit is taken in  $\text{PreStk}$ .*

*Remark.* Before we answer the “strong question”, let us note that it is *not* true that any  $\mathcal{X} \in \text{PreStk}_{\text{lft}}$ , viewed as a functor  $(\text{DGSch}_{\text{aft}})^{\text{op}} \rightarrow \infty\text{-Grpd}$ , is isomorphic to a left Kan extension along the embedding  $\text{DGSch}_{\text{aft}}^{\text{aff}} \hookrightarrow \text{DGSch}_{\text{aft}}^{\text{aff}}$  (or becomes such after sheafification). However, if  $\mathcal{X} \in \text{DGindSch}_{\text{lft}}$  admitted a presentation as a colimit of objects of  $\text{DGSch}_{\text{aft}}$ , it would automatically be such a left Kan extension. So, having such a property should be somewhat of a surprise. Nonetheless, the answer to the “strong” question is also affirmative:

**Proposition 1.6.4.** *For a DG indscheme  $\mathcal{X}$  locally almost of finite type there exists a filtered family*

$$A \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} : \alpha \mapsto X_\alpha,$$

*such that  $\mathcal{X}$  is isomorphic to*

$$(1.12) \quad \operatorname{colim}_{\alpha \in A} X_\alpha,$$

*where the colimit is taken in  $\mathrm{PreStk}$ .*

1.6.5. In fact, we shall prove a more precise version of the above assertions. Namely, in Sect. 3.5 we will prove:

**Proposition 1.6.6.** *Let  $\mathcal{X}$  be an object of  $\mathrm{DGindSch}_{\mathrm{laft}}$ .*

- (a) *The category  $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}$  in  $\mathcal{X}$  is filtered.*
- (b) *The natural map*

$$\operatorname{colim}_{Z \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X}} Z \rightarrow \mathcal{X},$$

*where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism.*

As a formal consequence, we obtain:

**Corollary 1.6.7.** *The following functors are cofinal:*

$$(1.13) \quad (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{qc-qs}})_{/\mathcal{X}}$$

$$(1.14) \quad (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{qc-qs}})_{\mathrm{closed}} \text{ in } \mathcal{X}$$

$$(1.15) \quad (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}$$

and

$$(1.16) \quad (<^\infty \mathrm{DGSch}_{\mathrm{ft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (<^\infty \mathrm{DGSch}_{\mathrm{ft}})_{/\mathcal{X}}.$$

**Corollary 1.6.8.** *An object  $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{Stk}$  belongs to  $\mathrm{DGindSch}_{\mathrm{laft}}$  if and only if:*

- *The category of closed embeddings  $Z \rightarrow \mathcal{X}$ , where  $Z \in \mathrm{DGSch}_{\mathrm{aft}}$ , is filtered.*
- *The functor (1.13) is cofinal.*

1.6.9. Note that Lemma 1.3.7 implies that for any presentation of  $\mathcal{X}$  as in Proposition 1.6.4, the tautological map

$$A \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X}$$

is cofinal.

## 2. SHEAVES ON DG INDSCHEMES

### 2.1. Quasi-coherent sheaves on a DG indscheme.

2.1.1. For any  $\mathcal{Y} \in \text{PreStk}$ , we have the symmetric monoidal category  $\text{QCoh}(\mathcal{Y})$  defined as in [GL:QCoh], Sect. 1.1.3. In particular, for  $\mathcal{X} \in \text{DGindSch}$  we obtain the symmetric monoidal category  $\text{QCoh}(\mathcal{X})$ .

If  $\mathcal{X}$  is written as (1.5), we have:

$$\text{QCoh}(\mathcal{X}) \simeq \lim_{\alpha} \text{QCoh}(X_{\alpha}),$$

where for  $\alpha_2 \geq \alpha_1$ , the map  $\text{QCoh}(X_{\alpha_2}) \rightarrow \text{QCoh}(X_{\alpha_1})$  is  $i_{\alpha_1, \alpha_2}^*$ . This follows from the fact that the functor

$$\text{QCoh}_{\text{PreStk}} : \text{PreStk}^{op} \rightarrow \text{DGCat}_{\text{cont}}$$

takes colimits in  $\text{PreStk}$  to limits in  $\text{DGCat}_{\text{cont}}$ .

Since the category  $\text{QCoh}(\mathcal{X})$  is given as a limit, it is not at all guaranteed that it will be compactly generated.

2.1.2. Recall the notion of a perfect object in  $\text{QCoh}(\mathcal{Y})$  for  $\mathcal{Y} \in \text{PreStk}$ , see, e.g., [GL:QCoh], Sect. 4.1.6.

The subcategory  $\text{QCoh}(\mathcal{Y})^{\text{perf}}$  coincides with that of dualizable objects of  $\text{QCoh}(\mathcal{Y})$  (see, e.g., *loc.cit.*, Lemma 4.2.2).

Also recall that if  $\mathcal{Y} = X$  is a quasi-compact and quasi-separated scheme, then the category  $\text{QCoh}(X)$  is compactly generated and  $\text{QCoh}(X)^c = \text{QCoh}(X)^{\text{perf}}$ . Moreover, the canonical self-duality

$$\text{QCoh}(X)^{\vee} \simeq \text{QCoh}(X)$$

can be described in either of the following two equivalent ways:

- The corresponding <sup>11</sup> functor  $\mathbb{D}_{\text{QCoh}(X)} : (\text{QCoh}(X)^c)^{op} \rightarrow \text{QCoh}(X)^c$  is the duality functor  $\mathcal{F} \mapsto \mathcal{F}^{\vee} : (\text{QCoh}(\mathcal{Y})^{\text{perf}})^{op} \rightarrow \text{QCoh}(\mathcal{Y})^{\text{perf}}$ .
- The pairing  $\text{QCoh}(X) \otimes \text{QCoh}(X) \rightarrow \text{Vect}$  is the composition

$$\text{QCoh}(X) \otimes \text{QCoh}(X) \xrightarrow{\cong} \text{QCoh}(X) \xrightarrow{\Gamma(X, -)} \text{Vect}.$$

Note that for an object  $\mathcal{Y} \in \text{PreStk}$  (and, in particular, for  $\mathcal{X} \in \text{DGindSch}$ ), the functor  $\Gamma(\mathcal{Y}, -) : \text{QCoh}(\mathcal{Y}) \rightarrow \text{Vect}$  is not, in general, continuous. Therefore, the functor

$$\text{QCoh}(\mathcal{Y}) \otimes \text{QCoh}(\mathcal{Y}) \xrightarrow{\cong} \text{QCoh}(\mathcal{Y}) \xrightarrow{\Gamma(\mathcal{Y}, -)} \text{Vect}$$

is not continuous either, and as such cannot serve as a candidate the duality pairing.

2.1.3. Let  $\mathcal{Y}$  be an arbitrary object of  $\text{PreStk}$ . We shall say that  $\mathcal{Y}$  is quasi-perfect if

- (i) The category  $\text{QCoh}(\mathcal{Y})$  is compactly generated.
- (ii) The compact objects of  $\text{QCoh}(\mathcal{Y})$  are *perfect*, and the duality functor

$$(2.1) \quad \mathcal{F} \mapsto \mathcal{F}^{\vee} : (\text{QCoh}(\mathcal{Y})^{\text{perf}})^{op} \simeq \text{QCoh}(\mathcal{Y})^{\text{perf}}$$

send  $(\text{QCoh}(\mathcal{Y})^c)^{op} \rightarrow \text{QCoh}(\mathcal{Y})^c$ .

Note that for  $\mathcal{Y}$  quasi-perfect, there exists a canonical equivalence  $\text{QCoh}(\mathcal{Y})^{\vee} \simeq \text{QCoh}(\mathcal{Y})$  given by the equivalence

$$\mathbb{D}_{\text{QCoh}(\mathcal{Y})} : (\text{QCoh}(\mathcal{Y})^c)^{op} \simeq \text{QCoh}(\mathcal{Y})^c$$

induced by the duality functor (2.1).

---

<sup>11</sup>Here and elsewhere, for a compactly generated category  $\mathbf{C}$ , we denote by  $\mathbb{D}_{\mathbf{C}}$  the canonical anti self-equivalence  $(\mathbf{C}^c)^{op} \rightarrow (\mathbf{C}^{\vee})^c$ .

The corresponding pairing  $\mathrm{QCoh}(\mathcal{Y}) \otimes \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{Vect}$  can be described as follows: it is obtained by ind-extending the pairing on compact objects given by

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(\mathcal{Y})^c \mapsto \Gamma(\mathcal{X}, \mathcal{F}_1 \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{F}_2) \in \mathrm{Vect}.$$

Indeed, this follows from the fact that for  $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$  and  $\mathcal{F}' \in \mathrm{QCoh}(\mathcal{X})$ , we have a functorial isomorphism

$$\mathrm{Hom}(\mathcal{F}^\vee, \mathcal{F}') \simeq \Gamma(\mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}').$$

Furthermore, note that  $\mathrm{QCoh}(\mathcal{X})^c$  is a monoidal ideal in  $\mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}}$ .

2.1.4. We shall see that certain DG indschemes are quasi-perfect in the above sense (see Sect. 7.2 and Sect. 10.2.1).

## 2.2. Ind-coherent sheaves on a DG indscheme.

2.2.1. Let  $\mathcal{Y} \in \mathrm{PreStk}$  be locally almost of finite type. In this case, following [GL:IndCoh], Sect. 9.1.4, we can introduce the category  $\mathrm{IndCoh}(\mathcal{Y})$ , which is a module category over  $\mathrm{QCoh}(\mathcal{Y})$ .

Indeed, by Sect. 9.1.7 of *loc.cit.*, we have a functor

$$\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^! : \mathrm{PreStk}_{\mathrm{laft}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

Let us denote by  $\mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}}^!$  the functor

$$(\mathrm{DGindSch}_{\mathrm{laft}})^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

obtained from  $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^!$  restriction along the fully faithful embedding

$$\mathrm{DGindSch}_{\mathrm{laft}} \hookrightarrow \mathrm{PreStk}_{\mathrm{laft}}.$$

Thus, for every  $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$ , we have a well-defined DG category  $\mathrm{IndCoh}(\mathcal{X})$ , which is a module for  $\mathrm{QCoh}(\mathcal{X})$ .

2.2.2. The above story also works when we replace the category  $\mathrm{PreStk}_{\mathrm{laft}}$  by  $\leq^n \mathrm{PreStk}_{\mathrm{lft}}$ , for a fixed  $n$  (see [GL:IndCoh], Sect. 9.1.6). Thus, we obtain a functor

$$\mathrm{IndCoh}_{\leq^n \mathrm{PreStk}_{\mathrm{lft}}}^! : \leq^n \mathrm{PreStk}_{\mathrm{lft}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

and by restriction, the functor

$$\mathrm{IndCoh}_{\leq^n \mathrm{DGindSch}_{\mathrm{lft}}}^! : \leq^n \mathrm{DGindSch}_{\mathrm{lft}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}.$$

Note, however, that the functor  $\mathrm{IndCoh}_{\leq^n \mathrm{PreStk}_{\mathrm{lft}}}^!$  can be expressed in terms of the functor  $\mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^!$  by the formula

$$\mathrm{IndCoh}_{\leq^n \mathrm{PreStk}_{\mathrm{lft}}}^! \simeq \mathrm{IndCoh}_{\mathrm{PreStk}_{\mathrm{laft}}}^! \circ \mathrm{LKE}_{\leq^n \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}} \hookrightarrow <^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}},$$

where  $\mathrm{LKE}_{\leq^n \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}} \hookrightarrow <^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}}$  denotes the functor

$$\leq^n \mathrm{PreStk}_{\mathrm{lft}} \rightarrow \mathrm{PreStk}_{\mathrm{laft}}$$

of left Kan extension along the embedding  $\leq^n \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}} \hookrightarrow <^\infty \mathrm{DGSch}_{\mathrm{ft}}^{\mathrm{aff}}$ , and which is the fully faithful left adjoint of the tautological restriction functor

$$\mathcal{Y} \mapsto \leq^n \mathcal{Y} : \mathrm{PreStk}_{\mathrm{laft}} \rightarrow \leq^n \mathrm{PreStk}_{\mathrm{lft}}.$$

**2.3. The convergence property of  $\text{IndCoh}$ .** Recall that the assignment

$$\mathcal{Y} \mapsto \text{IndCoh}(\mathcal{Y})$$

has the property that it only depends on the restriction  $\mathcal{Y}|_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}$ . In application to DG indschemes, this fact has several corollaries.

2.3.1. The first corollary is that for any  $\mathcal{X} \in \text{DGindSch}_{\text{laft}}$ , the category  $\text{IndCoh}(\mathcal{X})$  can be recovered from the values of  $\text{IndCoh}$  on the objects  ${}^{\leq n}\mathcal{X} \in {}^{\leq n}\text{DGindSch}_{\text{laft}}$ . Namely, we have the following assertion (Lemma 9.2.5 of [GL:IndCoh]) valid for any object of  $\text{PreStk}_{\text{laft}}$ .

**Lemma 2.3.2.** *For any  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$ , the natural map*

$$\text{IndCoh}_{\text{PreStk}_{\text{laft}}}^!(\mathcal{Y}) \rightarrow \lim_n \text{IndCoh}_{\leq n \text{PreStk}_{\text{laft}}}^!({}^{\leq n}\mathcal{Y})$$

*is an isomorphism.*

2.3.3. Let  $\mathcal{X}$  be an object of  $\text{DGindSch}_{\text{laft}}$ . For an integer  $n$ , consider the prestack

$${}^{\text{conv}, L}\tau^{\leq n}(\mathcal{X}) := {}^{\text{conv}}({}^L\tau^{\leq n}(\mathcal{X})) \in \text{PreStk}.$$

By Lemma 1.3.5, we have  ${}^{\text{conv}, L}\tau^{\leq n}(\mathcal{X}) \in \text{DGindSch}$ . We claim, in addition, that  ${}^{\text{conv}, L}\tau^{\leq n}(\mathcal{X})$ , in fact, belongs to  $\text{DGindSch}_{\text{laft}}$ .

More generally, we claim that for any  $\mathcal{Y}_n \in {}^{\leq n}\text{PreStk}_{\text{laft}}$ , which is  $k$ -truncated for some  $k$  (see [GL:Stacks], Sect. 1.7.7 for the notion of truncatedness), the object

$${}^{\text{conv}, L}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Y}_n) \in \text{PreStk}$$

belongs to  $\text{PreStk}_{\text{laft}}$ . This follows by unwinding the definitions and using Corollary 2.5.7 of [GL:Stacks].

**Proposition 2.3.4.** *For  $\mathcal{X} \in \text{DGindSch}_{\text{laft}}$ , the natural map*

$$\text{IndCoh}(\mathcal{X}) \rightarrow \lim_n \text{IndCoh}({}^{\text{conv}, L}\tau^{\leq n}(\mathcal{X}))$$

*is an equivalence.*

*Proof.* We shall prove this proposition for any  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$  such that for every  $n$ ,

$${}^{\leq n}\mathcal{Y} := \mathcal{Y}|_{\leq n \text{DGSch}^{\text{aff}}}$$

is  $k$ -truncated for some  $k$ ; by the above, for such a  $\mathcal{Y}$ , we have that  ${}^{\text{conv}, L}\tau^{\leq n}(\mathcal{Y}) \in \text{PreStk}_{\text{laft}}$ . Thus,  $\text{IndCoh}({}^{\text{conv}, L}\tau^{\leq n}(\mathcal{Y}))$  is well-defined.

By Lemma 2.3.2 and [GL:IndCoh], Corollary 9.4.3, it suffices to show that the map

$$\text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow <\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}({}^{\leq n}\mathcal{Y}|_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}}}) \rightarrow ({}^{\text{conv}, L}\tau^{\leq n}(\mathcal{Y}))|_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}$$

becomes an isomorphism after applying the functor  $L_{\text{laft}}$  (see [GL:Stacks], Sect. 2.6.1 for the notation).

We will show that for any  $\mathcal{Z} \in {}^{\leq n}\text{PreStk}_{\text{laft}}$ , the map

$$(2.2) \quad \begin{aligned} & \text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow <\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}(\mathcal{Z}) \rightarrow \\ & \rightarrow \left( {}^{\text{conv}, L}\text{LKE}_{\leq n \text{DGSch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}} \circ \text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow \leq n \text{DGSch}^{\text{aff}}}(\mathcal{Z}) \right)|_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \end{aligned}$$

becomes an isomorphism after applying the functor  $L_{\text{laft}}$ .

By definition,

$$({}^{\text{conv}}(-))|_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \simeq (-)|_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}},$$

and by [GL:Stacks], Lemma 2.6.3,

$$\left( {}^L(-) \right) |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \simeq L_{\text{laft}} \left( (-) |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \right).$$

Hence, the right-hand side in (2.2) is identified with

$$L_{\text{laft}} \left( \left( \text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Z}) \right) |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \right).$$

However,

$$\left( \text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{Z}) \right) |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \simeq \text{LKE}_{\leq n \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow <\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}(\mathcal{Z}),$$

so the required assertion follows from the fact that the natural map

$$L_{\text{laft}} \rightarrow L_{\text{laft}} \circ L_{\text{laft}}$$

is an isomorphism. □

2.3.5. As another corollary of the observation in Sect. 2.3 we obtain the following.

**Corollary 2.3.6.** *Let  $\mathcal{X} \in \text{DGindSch}$  be written as in (1.11). We have:*

$$\text{IndCoh}(\mathcal{X}) \simeq \lim_{\alpha} \text{IndCoh}(X_{\alpha}),$$

where for  $\alpha_2 \geq \alpha_1$ , the map  $\text{IndCoh}(X_{\alpha_2}) \rightarrow \text{IndCoh}(X_{\alpha_1})$  is  $i_{\alpha_1, \alpha_2}^!$ .

*Proof.* Note that for  $\mathcal{X}$  written as in (1.11), we have an isomorphism

$$\mathcal{X} |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}} \simeq \text{colim}_{\alpha \in A} X_{\alpha} |_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}},$$

where the colimit is taken in the category  $\text{PreStk}_{\text{laft}}$ .

The required assertion follows now from the fact that the functor  $\text{IndCoh}_{\text{PreStk}_{\text{laft}}}^!$  takes colimits in  $\text{PreStk}_{\text{laft}}$  to limits in  $\text{DGCat}_{\text{cont}}$ , by the definition of  $\text{IndCoh}_{\text{PreStk}_{\text{laft}}}^!$  as the right Kan extension of the functor  $\text{IndCoh}_{<\infty \text{DGSch}_{\text{ft}}^{\text{aff}}}^!$  along the Yoneda embedding

$$<\infty \text{DGSch}_{\text{ft}}^{\text{aff}} \hookrightarrow \text{PreStk}_{\text{laft}}.$$

□

**2.4. Interpretation of  $\text{IndCoh}$  as a colimit, self-duality and the t-structure.** One of the main advantages of the category  $\text{IndCoh}(\mathcal{X})$  over  $\text{QCoh}(\mathcal{X})$  for a DG indscheme  $\mathcal{X}$  is that the former admits an alternative description as a colimit.

2.4.1. Indeed, recall that for a closed embedding of DG schemes  $i : X_1 \rightarrow X_2$ , the functor

$$i^! : \text{IndCoh}(X_2) \rightarrow \text{IndCoh}(X_1)$$

admits a left adjoint,  $i_*^{\text{IndCoh}}$ , see [GL:IndCoh], Sect. 3.2.

By Corollary 2.3.6 and [GL:DG], Sect. 1.3.3, we have that for  $\mathcal{X}$  as in (1.11),

$$(2.3) \quad \text{IndCoh}(\mathcal{X}) \simeq \text{colim}_{\alpha} \text{IndCoh}(X_{\alpha}),$$

where for  $\alpha_2 \geq \alpha_1$ , the map  $\text{IndCoh}(X_{\alpha_2}) \rightarrow \text{IndCoh}(X_{\alpha_1})$  is  $(i_{\alpha_1, \alpha_2})_*^{\text{IndCoh}}$ .

In particular, by [GL:DG], Sect. 2.2.1, we obtain:

**Corollary 2.4.2.** *For  $\mathcal{X} \in \text{DGindSch}$ , the category  $\text{IndCoh}(\mathcal{X})$  is compactly generated.*

2.4.3. We shall now show that the category  $\mathrm{IndCoh}(\mathcal{X})$  is canonically self-dual, i.e.

$$(2.4) \quad \mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X}).$$

Let us write  $\mathcal{X}$  as in (1.11). Combining (2.3) with [GL:DG], Lemma 2.2.2 and [GL:IndCoh], Sect. 8.1.3 and 8.1.5 we obtain:

**Corollary 2.4.4.** *Serre duality defines a canonical equivalence:*

$$\mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X}).$$

Note that by Sect. 1.6.9, any other way of writing  $\mathcal{X}$  as in (1.12) will give rise to a canonically isomorphic duality functor.

2.4.5. Let us describe the equivalence of Corollary 2.4.4 more explicitly. Namely, we would like to describe the corresponding pairing:

$$(2.5) \quad \mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{Vect}.$$

2.4.6. For a DG scheme  $X$  almost of finite type, let

$$\Gamma^{\mathrm{IndCoh}}(X, -) : \mathrm{IndCoh}(X) \rightarrow \mathrm{Vect}$$

denote the functor  $p_*^{\mathrm{IndCoh}}$  of [GL:IndCoh], Proposition 3.1.1, where  $p : X \rightarrow \mathrm{pt}$ .

For a DG indscheme  $\mathcal{X}$ , written as in (1.11), we define the functor

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -) : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{Vect}$$

to be given by the compatible family of functors  $\Gamma^{\mathrm{IndCoh}}(X_\alpha, -) : \mathrm{IndCoh}(X_\alpha) \rightarrow \mathrm{Vect}$ .

Again, by Sect. 1.6.9, the above definition of  $\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)$  is canonically independent of the choice of the presentation (1.11).

2.4.7. The definition of the functor (2.4), along with [GL:IndCoh], Sect. 8.1.6 yields:

**Corollary 2.4.8.** *The functor (2.5) is canonically isomorphic to the composite*

$$\mathrm{IndCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\boxtimes} \mathrm{IndCoh}(\mathcal{X} \times \mathcal{X}) \xrightarrow{\Delta_{\mathcal{X}}^!} \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -)} \mathrm{Vect}.$$

2.4.9. *t-structure on IndCoh.*

For  $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$ , we define a t-structure on  $\mathrm{IndCoh}(\mathcal{X})$  as follows. An object

$$\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$$

belongs to  $\mathrm{IndCoh}^{\geq 0}$  if and only if for every closed embedding  $i : X \rightarrow \mathcal{X}$  with  $X \in \mathrm{DGSch}_{\mathrm{laft}}$ , the object  $i^!(\mathcal{F}) \in \mathrm{IndCoh}(X)$  belongs to  $\mathrm{IndCoh}(X)^{\geq 0}$ .

By construction, this t-structure is compatible with filtered colimits, i.e.,  $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$  is preserved by filtered colimits.

We can describe this t-structure and the category  $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$  more explicitly. Fix a presentation of  $\mathcal{X}$  as in (1.11). For each  $\alpha$ , let  $i_\alpha$  denote the corresponding map  $X_\alpha \rightarrow \mathcal{X}$ . By (2.3), we have a pair of adjoint functors

$$(i_\alpha)_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_\alpha) \rightleftarrows \mathrm{IndCoh}(\mathcal{X}) : i_\alpha^!.$$

**Lemma 2.4.10.** *Under the above circumstances we have:*

- (a) *An object  $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$  belongs to  $\mathrm{IndCoh}^{\geq 0}$  if and only if for every  $\alpha$ , the object  $i_\alpha^!(\mathcal{F}) \in \mathrm{IndCoh}(X_\alpha)$  belongs to  $\mathrm{IndCoh}(X_\alpha)^{\geq 0}$ .*
- (b) *The category  $\mathrm{IndCoh}(\mathcal{X})^{\leq 0}$  is generated under colimits by the essential images of the functors  $(i_\alpha)_*^{\mathrm{IndCoh}}(\mathrm{Coh}(X_\alpha)^{\leq 0})$ .*

*Proof.* It is easy to see that for a quasi-compact DG scheme  $X$ , the category  $\mathrm{IndCoh}(X)^{\leq 0}$  is generated under colimits by  $\mathrm{Coh}(^c X)^{\leq 0}$ . In particular, by adjunction, an object  $\mathcal{F} \in \mathrm{IndCoh}(X)$  is coconnective if and only if its restriction to  $^c X$  is coconnective.

Hence, in the definition of  $\mathrm{IndCoh}(\mathcal{X})^{\geq 0}$ , instead of all closed embeddings  $X \rightarrow \mathcal{X}$ , it suffices to use only those with  $X$  a classical scheme.

This implies point (a) of the lemma by Lemma 1.1.7. Point (b) follows formally from point (a). □

2.4.11. Suppose  $i : X \rightarrow \mathcal{X}$  is a closed embedding of a DG scheme into a DG indscheme. We then have:

**Lemma 2.4.12.** *The functor  $i_*^{\mathrm{IndCoh}}$  is t-exact.*

*Proof.* Since  $i_*^{\mathrm{IndCoh}}$  is the left adjoint of  $i^!$ , it is right t-exact. Thus we need to show that for  $\mathcal{F} \in \mathrm{IndCoh}(X)^{\geq 0}$ , we have  $i_\alpha^! \circ i_*^{\mathrm{IndCoh}}(\mathcal{F}) \in \mathrm{IndCoh}(X_\alpha)^{\geq 0}$  for every closed embedding  $i_\alpha : X_\alpha \rightarrow \mathcal{X}$ . By Proposition 1.6.6, we have

$$\mathcal{X} \simeq \operatorname{colim}_{X_\alpha \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X}} X_\alpha.$$

Therefore, by [GL:DG, Sect. 1.3.5], we have

$$i_\alpha^! \circ i_*^{\mathrm{IndCoh}}(\mathcal{F}) \simeq \operatorname{colim}_{i'_\beta : X \rightarrow X_\beta, i_{\alpha\beta} : X_\alpha \rightarrow X_\beta} i_{\alpha\beta}^! \circ (i'_\beta)_*^{\mathrm{IndCoh}}(\mathcal{F}),$$

where maps in the colimit are in  $(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X}$ . The lemma now follows from the fact that both  $i_{\alpha\beta}^!$  and  $(i'_\beta)_*^{\mathrm{IndCoh}}$  are left t-exact and the t-structure on  $\mathrm{IndCoh}$  is compatible with filtered colimits. □

## 2.5. Functoriality of $\mathrm{IndCoh}$ under pushforwards.

2.5.1. Recall the functor  $\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}} : \mathrm{DGSch}_{\mathrm{aft}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$  of [GL:IndCoh], Sect. 8.1.1, which assigns to  $X \in \mathrm{DGSch}_{\mathrm{aft}}$  the category  $\mathrm{IndCoh}(X)$  and to a map  $f : X_1 \rightarrow X_2$  the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_1) \rightarrow \mathrm{IndCoh}(X_2).$$

Let

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \subset (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}} \subset \mathrm{DGSch}_{\mathrm{aft}}$$

be the non-full subcategories, where we restrict 1-morphisms to be closed embeddings (resp., proper). Let

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}} \text{ and } \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}$$

be the restriction of  $\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}}$  to these subcategories.



2.5.2. Let

$$(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}} \subset (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}$$

denote the corresponding non-full subcategories of  $\mathrm{DGindSch}_{\mathrm{laft}}$ .

Recall that a map  $\mathcal{X}_1 \rightarrow \mathcal{X}_2$  of DG indschemes is called proper if and only if the corresponding map  ${}^{cl}\mathcal{X}_1 \rightarrow {}^{cl}\mathcal{X}_2$  is. The latter, by definition, means that whenever  $X_i \hookrightarrow \mathcal{X}_i$  are closed embeddings with  $X_i \in \mathrm{Sch}_{\mathrm{ft}}$  such that there exists a commutative diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & \mathcal{X}_1 \\ f' \downarrow & & \downarrow f \\ X_2 & \longrightarrow & \mathcal{X}_2, \end{array}$$

the map  $f'$  (which is automatically unique!), is proper.

Let

$$(2.6) \quad \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}}, \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}} \text{ and } \mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}}$$

denote the left Kan extensions of the functors

$$\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}}, \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}} \text{ and } \mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}}$$

along the fully faithful embeddings

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}, (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}$$

and

$$\mathrm{DGSch}_{\mathrm{aft}} \hookrightarrow \mathrm{DGindSch}_{\mathrm{laft}},$$

respectively.

From (2.3) and Sect. 1.6.9 we obtain:

**Corollary 2.5.3.** *For  $\mathcal{X} \in \mathrm{DGindSch}_{\mathrm{laft}}$ , the value of the functor  $\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}}$  on  $\mathcal{X}$  is canonically equivalent to  $\mathrm{IndCoh}(\mathcal{X})$ .*

2.5.4. By construction, we have the natural transformations

$$(2.7) \quad \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}} \rightarrow \mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}} \big|_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}} \text{ and } \\ \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}} \rightarrow \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}} \big|_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}}.$$

**Proposition 2.5.5.** *The natural transformations (2.7) are equivalences.*

*Proof.* For a given  $\mathcal{X} \in \mathrm{DGindSch}$ , the value of the functors (2.6) on it are given by

$$\mathrm{colim}_{X \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X}} \mathrm{IndCoh}(X), \quad \mathrm{colim}_{X \in (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}} \text{ over } \mathcal{X}} \mathrm{IndCoh}(X)$$

and

$$\mathrm{colim}_{X \in (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}} \mathrm{IndCoh}(X),$$

respectively.

Hence, to prove the proposition, it suffices to show that the functors

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}} \text{ over } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}$$

are cofinal. Since both arrows are fully faithful embeddings, it suffices to show that the functor

$$(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}} \text{ in } \mathcal{X} \rightarrow (\mathrm{DGSch}_{\mathrm{aft}})_{/\mathcal{X}}$$

is cofinal, but the latter is given by Corollary 1.6.7.  $\square$

2.5.6. Thus, from Proposition 2.5.5 we obtain that for a morphism  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  we have a well-defined functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}_1) \rightarrow \text{IndCoh}(\mathcal{X}_2).$$

Concretely, the functor  $f_*^{\text{IndCoh}}$  can be described as follows. By (2.3), objects of  $\text{IndCoh}(X)$  are colimits of objects of the form  $(i_1)_*^{\text{IndCoh}}(\mathcal{F}_1)$  for  $\mathcal{F}_1 \in \text{IndCoh}(X_1)$ , where  $X_1$  is a DG scheme almost of finite type equipped with a closed embedding  $X_1 \xrightarrow{i_1} \mathcal{X}_1$ . By continuity, the functor  $f_*^{\text{IndCoh}}$  is completely determined by its values on such objects.

By Corollary 1.6.7, we can factor the map

$$X_1 \xrightarrow{i_1} \mathcal{X}_1 \xrightarrow{f} \mathcal{X}_2$$

as

$$X_1 \xrightarrow{g} X_2 \xrightarrow{i_2} \mathcal{X}_2,$$

where  $X_2 \in \text{DGSch}_{\text{aft}}$  and  $i_2$  being a closed embedding. We set

$$f_*^{\text{IndCoh}}((i_1)_*^{\text{IndCoh}}(\mathcal{F}_1)) = (i_2)_*^{\text{IndCoh}}(g_*^{\text{IndCoh}}(\mathcal{F}_1)).$$

The content of Proposition 2.5.5 is that this construction extends to a well-defined functor  $f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}_1) \rightarrow \text{IndCoh}(\mathcal{X}_2)$ .

It is easy to see from [GL:IndCoh], Corollary 2.4.8 that the functor  $f_*^{\text{IndCoh}}$  is the dual of  $f^!$ .

Note that the functor  $\Gamma^{\text{IndCoh}}(\mathcal{X}, -)$  of Sect. 2.4.6 is a particular instance of this construction for  $\mathcal{X}_1 = \mathcal{X}$  and  $\mathcal{X}_2 = \text{pt}$ .

2.5.7. For a morphism of indschemes, the pushforward functor on  $\text{IndCoh}$  interacts with the t-structure in the usual way:

**Lemma 2.5.8.** *Let  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  be a map of indschemes. Then the functor  $f_*^{\text{IndCoh}}$  is left t-exact. Furthermore, if  $f$  is a closed embedding, then it is t-exact.*

*Proof.* Let  $\mathcal{F} \in \text{IndCoh}(\mathcal{X}_1)^{\geq 0}$ . We wish to show that  $f_*^{\text{IndCoh}}(\mathcal{F}) \in \text{IndCoh}(\mathcal{X}_2)^{\geq 0}$ . By Corollary 2.5.3 and Lemma 2.4.12, we can assume that  $\mathcal{F} = (i_1)_*^{\text{IndCoh}}(\mathcal{F}_1)$  for  $\mathcal{F}_1 \in \text{IndCoh}(X_1)^{\geq 0}$  where  $i_1 : X_1 \rightarrow \mathcal{X}_1$  is a closed embedding. Now, let

$$X_1 \xrightarrow{g} X_2 \xrightarrow{i_2} \mathcal{X}_2$$

be a factorization of  $f \circ i_1$ , where  $i_2$  is a closed embedding. We then have,

$$f_*^{\text{IndCoh}}(\mathcal{F}) \simeq f_*^{\text{IndCoh}}(i_1)_*^{\text{IndCoh}}(\mathcal{F}_1) = (i_2)_*^{\text{IndCoh}}(g_*^{\text{IndCoh}}(\mathcal{F}_1)).$$

By Lemma 2.4.12,  $(i_2)_*^{\text{IndCoh}}(g_*^{\text{IndCoh}}(\mathcal{F}_1)) \in \text{IndCoh}(\mathcal{X}_2)^{\geq 0}$ .

Now, suppose that  $f$  is a closed embedding. In this case, we wish to show that  $f_*^{\text{IndCoh}}$  is also right exact. Let  $\mathcal{F} \in \text{IndCoh}(\mathcal{X}_1)^{\leq 0}$ . By Lemma 2.4.10(b), we can assume that  $\mathcal{F} = (i_1)_*^{\text{IndCoh}}(\mathcal{F}_1)$  for  $\mathcal{F}_1 \in \text{IndCoh}(X_1)^{\leq 0}$  where  $i_1 : X_1 \rightarrow \mathcal{X}_1$  is a closed embedding. The result now follows from the fact that the composed map

$$X_1 \rightarrow \mathcal{X}_1 \rightarrow \mathcal{X}_2$$

is a closed embedding and Lemma 2.4.12. □

2.5.9. Consider the functor

$$\mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}}^! : \mathrm{DGindSch}_{\mathrm{laft}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}},$$

and let

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}}^! \text{ and } \mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}}^!$$

be the restrictions of  $\mathrm{IndCoh}_{\mathrm{DGindSch}_{\mathrm{laft}}}^!$  to the corresponding subcategories.

In addition, consider the corresponding functors

$$\mathrm{IndCoh}_{\mathrm{DGSch}_{\mathrm{aft}}}^!, \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}^! \text{ and } \mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}}^!$$

for  $\mathrm{DGSch}_{\mathrm{aft}}$  instead of  $\mathrm{DGindSch}_{\mathrm{laft}}$ .

As in Proposition 2.5.5, we have:

**Lemma 2.5.10.** *The natural maps*

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}}^! \rightarrow \mathrm{RKE}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}^{\mathrm{op}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}^{\mathrm{op}}} (\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}^!)$$

and

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}}^! \rightarrow \mathrm{RKE}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}^{\mathrm{op}} \hookrightarrow (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{closed}}^{\mathrm{op}}} (\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}}^!)$$

are isomorphisms.

We shall now deduce the following:

**Corollary 2.5.11.** *The functor*

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}} : (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

is obtained from the functor

$$\mathrm{IndCoh}_{(\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}}^! : (\mathrm{DGindSch}_{\mathrm{laft}})_{\mathrm{proper}}^{\mathrm{op}} \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$$

by passing to left adjoints.

This corollary means that for a proper map  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  in  $\mathrm{DGindSch}_{\mathrm{laft}}$ , the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}_1) \rightarrow \mathrm{IndCoh}(\mathcal{X}_2)$$

is the left adjoint of  $f^! : \mathrm{IndCoh}(\mathcal{X}_2) \rightarrow \mathrm{IndCoh}(\mathcal{X}_1)$  in a coherent way.

*Proof.* This follows from the corresponding fact for the functors  $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}$  and  $\mathrm{IndCoh}_{(\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{proper}}}^!$  (see [GL:IndCoh], Theorem 5.2.2), and the following general assertion:

Let  $F : \mathbf{C}_1 \rightarrow \mathbf{C}_2$  be a functor between  $\infty$ -categories. Let  $\Phi_1 : \mathbf{C}_1 \rightarrow \mathrm{DGCat}_{\mathrm{cont}}$  be a functor such that for every  $\mathbf{c}'_1 \rightarrow \mathbf{c}''_1$ , the corresponding functor

$$\Phi_1(\mathbf{c}'_1) \rightarrow \Phi_1(\mathbf{c}''_1)$$

admits a right adjoint. Let  $\Psi_1 : \mathbf{C}_1^{\mathrm{op}} \rightarrow \mathrm{DGCat}$  be the resulting functor given by taking the right adjoints.

Let  $\Phi_2$  and  $\Psi_2$  be the left (resp., right) Kan extension of  $\Phi_1$  (resp.,  $\Psi_1$ ) along  $F$  (resp.,  $F^{\mathrm{op}}$ ).

The following is a version of [GL:DG], Lemma 1.3.3:

**Lemma 2.5.12.** *Under the above circumstances, the functor  $\Psi_2$  is obtained from  $\Phi_2$  by taking right adjoints.*

□

2.5.13. Let

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ f_Y \downarrow & & \downarrow f_X \\ \mathcal{Y}_2 & \xrightarrow{g_2} & \mathcal{X}_2 \end{array}$$

be a Cartesian diagram of DG indschemes, with the maps  $f_X$  and  $f_Y$  proper. By adjunction, we obtain a natural transformation

$$(2.8) \quad (f_Y)_*^{\mathrm{IndCoh}} \circ g_1^! \rightarrow g_2^! \circ (f_X)_*^{\mathrm{IndCoh}}.$$

**Proposition 2.5.14.** *The natural transformation (2.8) is an isomorphism.*

*Proof.* The assertion readily reduces to the case when  $\mathcal{Y}_2 = Y_2$  is a DG scheme. By (2.3), we can assume that  $\mathcal{X}_1 = X_1$  is also a DG scheme. Factoring the map  $X_1 \rightarrow X_2$  as a composition

$$X_1 \rightarrow X_2 \rightarrow \mathcal{X}_2,$$

where the first arrow is a proper map of DG schemes, and the second arrow is a closed embedding, and using the fact that the assertion holds for maps of schemes (see [GL:IndCoh, Proposition 3.2.7]), we are reduced to the case when  $f_X : X_1 \rightarrow X_2$  is a closed embedding.

Writing

$$\mathcal{X}_2 = \operatorname{colim}_{\alpha} X_{2,\alpha}$$

as in (1.12), we can assume that both maps

$$Y_2 \rightarrow \mathcal{X}_2 \text{ and } X_1 \rightarrow \mathcal{X}_2$$

factor through maps

$$Y_2 \xrightarrow{g_{2,\alpha}} X_{2,\alpha} \text{ and } X_1 \xrightarrow{f_{X,\alpha}} X_{2,\alpha},$$

respectively for all  $\alpha$ .

By [GL:DG], Sect. 1.3.5., we have

$$g_2^! \circ (f_X)_*^{\mathrm{IndCoh}} \simeq \operatorname{colim}_{\alpha} g_{2,\alpha}^! \circ (f_{X,\alpha})_*^{\mathrm{IndCoh}}.$$

Furthermore,

$$\mathcal{Y}_1 \simeq \operatorname{colim}_{\alpha} (Y_2 \times_{X_{2,\alpha}} X_1),$$

and the functor  $(f_Y)_*^{\mathrm{IndCoh}} \circ g_1^!$  is given by

$$\operatorname{colim}_{\alpha} (f_{Y,\alpha})_*^{\mathrm{IndCoh}} \circ g_{1,\alpha}^!,$$

where  $f_{X,\alpha}$  and  $g_{1,\alpha}$  are as in the following Cartesian diagram:

$$\begin{array}{ccc} Y_2 \times_{X_{2,\alpha}} X_1 & \xrightarrow{g_{1,\alpha}} & X_1 \\ f_{Y,\alpha} \downarrow & & \downarrow f_{X,\alpha} \\ Y_2 & \xrightarrow{g_{2,\alpha}} & X_{2,\alpha}. \end{array}$$

Hence, the proposition follows from the fact that the maps

$$(f_{Y,\alpha})_*^{\mathrm{IndCoh}} \circ g_{1,\alpha}^! \rightarrow g_{2,\alpha}^! \circ (f_{X,\alpha})_*^{\mathrm{IndCoh}}$$

are isomorphisms. □

*Remark 2.5.15.* The isomorphisms (2.8) can be defined for all maps  $f_X$ , and not just proper ones. However, the construction is more involved as there is no a priori map in either direction. For an individual  $f$ , such an isomorphism is easy to deduce from [GL:IndCoh], Sect. 5, where the corresponding natural transformations were constructed in the case of DG schemes. However, a functorial construction of these natural transformations for indschemes compatible with composition requires additional work and will be carried out in [GL:Funct].

2.5.16. Recall the category of correspondences  $(\text{PreStk}_{\text{laft}})_{\text{corr:sch-qc;all}}$  of [GL:IndCoh], Sect. 9.6.1. Similarly, consider the category  $(\text{PreStk}_{\text{laft}})_{\text{corr:indsch+proper;all}}$ .

As in *loc.cit.*, Proposition 9.6.3, Proposition 2.5.14 allows to combine the functors

$$\text{IndCoh}_{\text{PreStk}_{\text{laft}}}^! : (\text{PreStk}_{\text{laft}})^{op} \rightarrow \text{DGCat}_{\text{cont}}$$

and

$$\text{IndCoh}_{\text{DGindSch}_{\text{laft}}} : \text{DGindSch}_{\text{laft}} \rightarrow \text{DGCat}_{\text{cont}}$$

to a functor

$$\text{IndCoh}_{(\text{PreStk}_{\text{laft}})_{\text{corr:indsch+proper;all}}} : (\text{PreStk}_{\text{laft}})_{\text{corr:indsch+proper;all}} \rightarrow \text{DGCat}_{\text{cont}}.$$

Informally, for every morphism  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  in  $\text{PreStk}_{\text{laft}}$ , which is ind-schematic and proper (i.e., one whose base change by an object  $S \in \text{DGSch}_{\text{aff}}^{\text{aff}}$  yields a DG indscheme that maps properly to  $S$ ), we can define the functor

$$f_*^{\text{IndCoh}} : \text{IndCoh}(\mathcal{X}_1) \rightarrow \text{IndCoh}(\mathcal{X}_2),$$

which satisfies base change with respect to  $!$ -pullbacks under  $g_2 : \mathcal{Y}_2 \rightarrow \mathcal{X}_2$  along the diagram

$$\begin{array}{ccc} \mathcal{Y}_1 & \xrightarrow{g_1} & \mathcal{X}_1 \\ f' \downarrow & & \downarrow f \\ \mathcal{Y}_2 & \xrightarrow{g_2} & \mathcal{X}_2. \end{array}$$

In [GL:Funct], we will extend the functor  $\text{IndCoh}$  to a larger category of correspondences which contains both  $(\text{PreStk}_{\text{laft}})_{\text{corr:sch-qc;all}}$  and  $(\text{PreStk}_{\text{laft}})_{\text{corr:indsch+proper;all}}$ .

## 2.6. Groupoids in DGindSch.

2.6.1. Let  $\mathcal{X}^\bullet$  be a simplicial object in  $\text{DGindSch}$ , arising from a groupoid object

$$(2.9) \quad p_s, p_t : \mathcal{X}^1 \rightrightarrows \mathcal{X}^0$$

(see [Lu0], Definition 6.1.2.7).

Suppose that the face maps in the above simplicial DG indscheme are proper (equivalently, the maps  $p_s, p_t$  in (2.9) are proper).

In this case, the forgetful functor

$$\text{Tot}(\text{IndCoh}(\mathcal{X}^\bullet)) \rightarrow \text{IndCoh}(\mathcal{X}^0)$$

admits a left adjoint; moreover, the resulting monad on  $\text{IndCoh}(\mathcal{X}_0)$ , when viewed as a plain endo-functor of  $\text{IndCoh}(\mathcal{X}_0)$ , is naturally isomorphic to

$$(p_t)_*^{\text{IndCoh}} \circ (p_s)^!.$$

The proof is the same as that of [GL:IndCoh, Proposition 7.2.2].

2.6.2. Assume that in the situation of Sect. 2.6.1, the groupoid arises as the Čech nerve of a morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$ , which is proper and surjective.<sup>12</sup> Let  $\mathcal{X}^\bullet/\mathcal{Y}$  denote the resulting simplicial object.

In this case, the augmentation

$$\mathcal{X}^\bullet/\mathcal{Y} \rightarrow \mathcal{Y}$$

gives rise to a functor

$$(2.10) \quad \mathrm{IndCoh}(\mathcal{Y}) \rightarrow \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet/\mathcal{Y})).$$

As in [GL:IndCoh, Proposition 7.2.2] we have:

**Lemma 2.6.3.** *Under the above circumstances, the functor (2.10) is an equivalence.*

Note that the composition

$$\mathrm{IndCoh}(\mathcal{Y}) \rightarrow \mathrm{Tot}(\mathrm{IndCoh}(\mathcal{X}^\bullet/\mathcal{Y})) \rightarrow \mathrm{IndCoh}(\mathcal{X})$$

is the functor  $f^!$ , and hence its left adjoint is  $f_*^{\mathrm{IndCoh}}$ .

### 3. CLOSED EMBEDDINGS INTO A DG INDScheme AND PUSH-OUTS

Let  $X$  be a scheme, and  $Z_1$  and  $Z_2$  be two closed subschemes. In this case, we have a well-defined operation of taking the union of  $Z_1$  and  $Z_2$  in  $X$ ; in fact, this is the coproduct in the category of closed subschemes of  $X$  (locally, the ideal of the union is the intersection of the ideals of  $Z_1$  and  $Z_2$ ). The same operation is well-defined when  $X$  is no longer a scheme, but an indscheme: indeed the union of  $Z_1$  and  $Z_2$  in  $X$  is the same as their union in  $X'$ , if  $X'$  is another closed subscheme of  $X$  which contains  $Z_1$  and  $Z_2$ .

However, one might be suspicious of the operation of union in the DG setting: indeed, closed DG subschemes are no longer in bijection with “ideals.”

The goal of this section is to show that as far as the operation of union is concerned, things work as usual.

In addition, we will consider a particular paradigm where push-outs in the category of DG schemes exist and are well-behaved. This will allow us, in particular, to show that DG ind-schemes contain “many” closed subschemes.

#### 3.1. Closed embeddings into a DG scheme.

3.1.1. For a morphism  $f : Y \rightarrow X$  in  $\mathrm{DGSch}$  consider the category of its factorizations

$$Y \rightarrow Z \xrightarrow{\phi} X,$$

where  $\phi$  is a closed embedding. Morphisms in this category are commutative diagrams

$$(3.1) \quad \begin{array}{ccccc} & & Z_1 & & \\ & \nearrow & \downarrow & \searrow \phi_1 & \\ Y & & & & X \\ & \searrow & \downarrow & \nearrow \phi_2 & \\ & & Z_2 & & \end{array}$$

Denote this category  $\mathrm{DGSch}_{Y/\mathrm{closed}} \text{ in } X$ .

<sup>12</sup>I.e., the base change of  $f$  by an object of  $\mathrm{DGSch}_{\mathrm{aft}}^{\mathrm{aff}}$  yields a morphism surjective on  $k$ -points.

3.1.2. We shall prove:

**Proposition 3.1.3.** *The category  $\mathrm{DGSch}_{Y/\mathrm{closed}} \text{ in } X$  contains coproducts.*

As a formal corollary, we obtain:

**Corollary 3.1.4.** *The category  $\mathrm{DGSch}_{Y/\mathrm{closed}} \text{ in } X$  contains push-outs.*

*Proof.* For a diagram

$$(3.2) \quad \begin{array}{ccccc} & & Z_1 & & \\ & \nearrow & & \searrow \phi_1 & \\ Y & \rightarrow & Z & \xrightarrow{\phi} & X \\ & \searrow & & \nearrow \phi_2 & \\ & & Z_2 & & \end{array}$$

the push-out  $Z_1 \sqcup_Z Z_2$  in  $\mathrm{DGSch}_{Y/\mathrm{closed}} \text{ in } X$  is isomorphic to the coproduct of  $Z_1$  and  $Z_2$  in  $\mathrm{DGSch}_{Z/\mathrm{closed}} \text{ in } X$ .  $\square$

**Corollary 3.1.5.** *For an  $X \in \mathrm{DGSch}$ , the category of closed embeddings  $Z \rightarrow X$  contains push-outs and coproducts.*

**Corollary 3.1.6.** *For an  $X \in \mathrm{DGSch}$ , the category of closed embeddings  $Z \rightarrow X$  is filtered.*

Note that the last corollary is point (a) of Proposition 1.4.3.

3.1.7. *Proof of Proposition 3.1.3.* Let

$$(3.3) \quad \begin{array}{ccccc} & & Z_1 & & \\ & \nearrow \psi_1 & & \searrow \phi_1 & \\ Y & & & & X \\ & \searrow \psi_2 & & \nearrow \phi_2 & \\ & & Z_2 & & \end{array}$$

be two objects in  $\mathrm{DGSch}_{Y/\mathrm{closed}} \text{ in } X$ .

The question of the existence of their coproduct is Zariski local on  $X$ . So, without loss of generality, let  $X = \mathrm{Spec}(A)$  and  $Z_i = \mathrm{Spec}(B_i)$ . The maps  $\psi_i$  are given by algebra homomorphisms  $B_i \rightarrow \Gamma(Y, \mathcal{O}_Y)$ . Consider the (not necessarily connective algebra)

$$B_1 \times_{\Gamma(Y, \mathcal{O}_Y)} B_2,$$

and set

$$C' := \tau^{\leq 0} \left( B_1 \times_{\Gamma(Y, \mathcal{O}_Y)} B_2 \right).$$

We have a homomorphism  $A \rightarrow C'$ . Let  $C$  be the union of connected components of  $C'$  corresponding to the image of

$${}^{cl}A \rightarrow {}^{cl}C'.$$

The sought-for coproduct is  $\mathrm{Spec}(C)$ .

□

3.1.8. Let  $(Z_1 \sqcup_Y Z_2)_{\mathrm{closed in } X}$  denote the coproduct of  $Z_1$  and  $Z_2$  in  $\mathrm{DGSch}_{Y/\mathrm{closed in } X}$ . We shall simply write  $(Z_1 \sqcup Z_2)_{\mathrm{closed in } X}$  if  $Y = \emptyset$ .

As a corollary (of the proof) we obtain:

**Corollary 3.1.9.** *Let  $X \rightarrow X'$  be a closed embedding. Then the natural map*

$$(Z_1 \sqcup_Y Z_2)_{\mathrm{closed in } X'} \rightarrow (Z_1 \sqcup_Y Z_2)_{\mathrm{closed in } X}$$

*is an isomorphism.*

3.1.10. Note that it follows from the construction that if  $Z_1$ ,  $Z_2$  and  $Y$  are  $n$ -coconnective, then so is  $(Z_1 \sqcup_Y Z_2)_{\mathrm{closed in } X}$ .

In addition, we have:

**Corollary 3.1.11.** *Suppose that  $f : Y \rightarrow X$  is affine (resp., of cohomological amplitude  $k$  for the  $f_*$  functor). Then the map of truncations*

$$L_{\tau \leq n} \left( \left( L_{\tau \leq m}(Z_1) \underset{L_{\tau \leq m}(Y)}{\sqcup} L_{\tau \leq m}(Z_2) \right)_{\mathrm{closed in } X} \right) \rightarrow L_{\tau \leq n} \left( (Z_1 \sqcup_Y Z_2)_{\mathrm{closed in } X} \right)$$

*is an isomorphism, whenever  $m \geq n + 1$  (resp.,  $m \geq n + 1 + k$ ).*

3.1.12. Let us note that the proof of Proposition 3.1.3 also shows the following:

**Corollary 3.1.13.** *The category  $\mathrm{DGSch}_{Y/\mathrm{closed in } X}$  has an initial object.*

We will refer to the above initial object as *the closure of the image of  $f$* , and denote it by  $\overline{\mathrm{Im}}(f)$ . The conclusions of Corollary 3.1.9 and Corollary 3.1.11 apply to the formation of  $\overline{\mathrm{Im}}(f)$  as well.

## 3.2. The case of DG indschemes.

3.2.1. Recall the notion of a closed embedding for a morphism in  $\mathrm{PreStk}$ , see Sect. 1.4.1.

Let  $\mathcal{X}$  be an object of  $\mathrm{DGindSch}$ , and let us write  ${}^{cl}\mathcal{X}$  as in (1.1). Let  $Y \in \mathrm{DGSch}_{\mathrm{qc-qc}}$  and let  $f : Y \rightarrow \mathcal{X}$  be a map. It is easy to see that  $f$  is a closed embedding in the above sense if and only if for any/some index  $\alpha$  as in Lemma 1.1.7 for which  ${}^{cl}f : {}^{cl}Y \rightarrow {}^{cl}\mathcal{X}$  factors as

$$(3.4) \quad {}^{cl}f_{\alpha} : {}^{cl}Y \rightarrow X_{\alpha},$$

where the map  ${}^{cl}f_{\alpha}$  is a closed embedding.

3.2.2. For  $\mathcal{X} \in \mathrm{DGindSch}$ ,  $Y \in \mathrm{DGSch}_{\mathrm{qc-qc}}$  and a morphism  $Y \rightarrow \mathcal{X}$ , let  $\mathrm{DGSch}_{Y/\mathrm{closed in } \mathcal{X}}$  denote the category of factorizations

$$Y \rightarrow Z \xrightarrow{\phi} \mathcal{X},$$

where  $\phi$  is a closed embedding.

**Proposition 3.2.3.** *The category  $\mathrm{DGSch}_{Y/\mathrm{closed in } \mathcal{X}}$  contains coproducts.*

This proposition has corollaries similar to those of Proposition 3.1.3:

**Corollary 3.2.4.** *The category  $\mathrm{DGSch}_{Y/\mathrm{closed in } \mathcal{X}}$  contains push-outs.*



**Corollary 3.2.5.** *For  $\mathcal{X} \in \text{DGindSch}$ , the category of closed embeddings  $Z \rightarrow \mathcal{X}$ , where  $Z \in \text{DGSch}_{\text{qc-qs}}$ , contains push-outs and coproducts.*

**Corollary 3.2.6.** *For  $\mathcal{X} \in \text{DGindSch}$ , the category of closed embeddings  $Z \rightarrow \mathcal{X}$ , where  $Z \in \text{DGSch}_{\text{qc-qs}}$ , is filtered.*

*Proof.* (of Proposition 3.2.3)

*Remark.* As Proposition 3.2.3 will be used in the proof of Proposition 1.3.2, we will not be able to use the existence of a presentation as in (1.5).

Assume first that  $Y$ ,  $Z_1$  and  $Z_2$  are eventually coconnective, i.e.,  $n$ -coconnective for some  $n$ . Then we can work in the categories  $\leq^n \text{DGSch}^{\text{aff}}$  and  $\leq^n \text{DGindSch}$ . We replace  $\mathcal{X}$  by  $\leq^n \mathcal{X}$ , and representing it as in (1.1), we obtain that the statement follows from Corollary 3.1.9.

Let  $k$  denote the cohomological amplitude of the functor

$$({}^{cl}f_\alpha)_* : \text{QCoh}({}^{cl}Y) \rightarrow \text{QCoh}(X_\alpha),$$

where  ${}^{cl}f_\alpha$  is as in (3.4).

For an integer  $n$ , consider the DG scheme

$$\left( (Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}} \right)_n := {}^{L_{\mathcal{T}} \leq n} \left( \left( {}^{L_{\mathcal{T}} \leq m}(Z_1) \sqcup_{{}^{L_{\mathcal{T}} \leq m}(Y)} {}^{L_{\mathcal{T}} \leq m}(Z_2) \right)_{\text{closed in } \mathcal{X}} \right)$$

for any  $m \geq n + 1 + k$ . Note that for  $n_1 \leq n_2$ , we have

$$(3.5) \quad {}^{L_{\mathcal{T}} \leq n_1} \left( \left( (Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}} \right)_{n_2} \right) \simeq \left( (Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}} \right)_{n_1},$$

by Corollary 3.1.11.

The required push-out

$$(Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}}$$

is a DG scheme equal to

$$\text{colim}_n \left( (Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}} \right)_n,$$

where the colimit is taken in the category of DG schemes, which is well-defined due to (3.5).  $\square$

3.2.7. For future reference, we note that the above proof shows that the contents of Sect. 3.1.12 carry over to the situation where the target is a DG indscheme  $\mathcal{X}$  (instead of a DG scheme).

3.2.8. Also, note that if, in the situation of Proposition 3.2.3, the DG schemes  $Y$ ,  $Z_1$  and  $Z_2$  are almost of finite type, and  $Y \rightarrow \mathcal{X}$  is proper (i.e., the corresponding map at the classical level is proper), then  $(Z_1 \sqcup_Y Z_2)_{\text{closed in } \mathcal{X}}$  is also almost of finite type.

3.3. **A digression on push-outs.** Let

$$(3.6) \quad \begin{array}{ccc} & & Z_1 \\ & \nearrow f_1 & \\ Z & & \\ & \searrow f_2 & \\ & & Z_2 \end{array}$$

be a diagram in  $\text{DGSch}$ .

We wish to consider the push-out of this diagram in  $\text{DGSch}$ . Note that push-outs of (DG) schemes are not among the standard practices in algebraic geometry; this operation is in general quite ill-behaved unless we impose some particular conditions on morphisms under which we are taking push-outs. In what follows we will consider three rather special situations where push-outs are manageable.

3.3.1. First, assume that the maps  $f_1$  and  $f_2$  are closed embeddings.

We claim that in this case the push-out  $Z_1 \sqcup_Z Z_2 \in \text{DGSch}$  exists, and can be described as follows.

3.3.2. First, assume that the DG schemes involved are affine  $Z_i = \text{Spec}(A_i)$ ,  $Z = \text{Spec}(A)$ . Note that since the homomorphisms  ${}^{cl}A_i \rightarrow {}^{cl}A$  are surjective, the algebra  $A_1 \times_A A_2$  is connective. In fact, for the algebra  $A_1 \times_A A_2$  to be connective, it suffices that one of the homomorphisms  ${}^{cl}A_i \rightarrow {}^{cl}A$  be surjective.

Consider

$$Z' := \text{Spec}(A_1 \times_A A_2).$$

The maps  $Z_i \rightarrow Z'$  are closed embeddings, and it is easy to see that the Zariski topology on  $Z'$  is the quotient of that on  $Z_1 \sqcup Z_2$ .

It is easy to see that the above maps  $Z_i \rightarrow Z'$  identify  $Z'$  with the sought-for push-out.

*Warning:* Note that if one of the maps  $f_i$  fails to be a closed embeddings, it is no longer true that the push-out in the category of affine DG schemes is a push-out in the category of schemes. A counter-example is

$$\mathbb{A}^1 \times (\mathbb{A}^1 - 0) \hookleftarrow \{0\} \times (\mathbb{A}^1 - 0) \hookrightarrow \{0\} \times \mathbb{A}^1.$$

3.3.3. In the general case the push-out  $Z_1 \sqcup_Z Z_2$  can be described as follows: we can find open affine coverings  $U_1^\alpha, \alpha \in A$  of  $Z_1$  and  $U_2^\beta, \beta \in B$  of  $Z_2$  such that

$$U_1^\alpha \cap Z = U_{\alpha,\beta} = U_2^\beta \cap Z.$$

In this case, the sought-for push-out is covered by the affine schemes

$$U_\alpha^1 \sqcup_{U_{\alpha,\beta}} U_\beta^2.$$

3.3.4. Let us now replace the condition that both maps  $f_i$  be closed embeddings by the following one: we shall require that  $f_1$  be affine, and that  $f_2$  be a closed embedding and induce an isomorphism  ${}^{cl,red}Z \rightarrow {}^{cl,red}Z_2$ .<sup>13</sup>

In this case, it is easy to see that the construction of  $Z_1 \sqcup_Z Z_2$  described above still works. Moreover, the resulting map

$$Z_1 \rightarrow Z_1 \sqcup_Z Z_2$$

is still a closed embedding, which induces an isomorphism at the level of the corresponding reduced classical schemes.

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<sup>13</sup>Here and elsewhere, for a DG scheme  $X$ , we denote by  ${}^{cl,red}X$  the underlying reduced classical scheme.

3.3.5. We shall now generalize the situation of Sect. 3.3.4 even further: we shall assume that the map  $f_1 : Z \rightarrow Z_1$  is of finite cohomological amplitude  $k$  for the  $(f_1)_*$  functor, and that  $f_2 : Z \rightarrow Z_2$  is a closed embedding which induces an isomorphism  $\leq^{k+n} Z \rightarrow \leq^k Z_2$  for some  $n \geq 0$ .

Again, we claim that in this case the above construction of  $Z_1 \sqcup_Z Z_2$  works. Moreover, the resulting map

$$Z_1 \rightarrow Z_1 \sqcup_Z Z_2$$

is still a closed embedding that induces an isomorphism  $\leq^n Z_1 \simeq \leq^n (Z_1 \sqcup_Z Z_2)$ .

3.3.6. We note that by construction, if in the above situation the DG schemes  $Z_1$ ,  $Z_2$  and  $Z$  are quasi-compact and quasi-separated, then so is  $Z_1 \sqcup_Z Z_2$ .

3.3.7. Similarly, if  $Z_1$ ,  $Z_2$  and  $Z$  are (locally) almost of finite type, then so is their push-out, provided that the map  $Z \rightarrow Z_1$  is finite (i.e., the corresponding map of the underlying classical schemes is finite).

**3.4. Compatibility with push-outs.** Let us observe the following property enjoyed by ind-schemes:

**Proposition 3.4.1.** *Let*

$$\begin{array}{ccc} Z & \longrightarrow & Z_1 \\ \downarrow & & \downarrow \\ Z_2 & \longrightarrow & Z' \end{array}$$

*be a push-out diagram in  $\mathrm{DGSch}_{\mathrm{qc-qc}}$ , where  $Z, Z_1, Z_2$  are eventually coconnective. Then for  $\mathcal{X} \in \mathrm{DGindSch}$ , the natural map*

$$\mathrm{Maps}(Z', \mathcal{X}) \rightarrow \mathrm{Maps}(Z_1, \mathcal{X}) \times_{\mathrm{Maps}(Z, \mathcal{X})} \mathrm{Maps}(Z_2, \mathcal{X})$$

*is an isomorphism.*

*Proof.* Suppose that  $Z, Z_1, Z_2$  are  $n$ -coconnective. By adjunction, we obtain that  $Z'$  is  $n$ -coconnective as well.

The assertion of the proposition now follows from Lemma 1.1.7 and the fact that Cartesian products commute with filtered colimits.  $\square$

*Remark.* In the above proposition we had to make the eventual coconnectivity assumption, because it will be used for the proof of Proposition 1.3.2. However, assuming this proposition, and hence, Lemma 1.3.7, we will be able to prove the same assertion for any  $Z, Z_1, Z \in \mathrm{DGSch}_{\mathrm{qc-qc}}$ . The next corollary, which will be also used in the proof of Proposition 1.3.2, gives a partial result along these lines:

**Corollary 3.4.2.** *Let  $Z_1 \xleftarrow{f_1} Z \xrightarrow{f_2} Z_2$  be a push-out diagram in  $\mathrm{DGSch}_{\mathrm{qc-qc}}$  such that either both  $f_1$  and  $f_2$  are closed embeddings, or they fall into the paradigm of Sections 3.3.4 or 3.3.5. Then for the push-out  $Z' := Z_1 \sqcup_Z Z_2$  in  $\mathrm{DGSch}$ , and  $\mathcal{X} \in \mathrm{DGindSch}$ , the natural map*

$$\mathrm{Maps}(Z', \mathcal{X}) \rightarrow \mathrm{Maps}(Z_1, \mathcal{X}) \times_{\mathrm{Maps}(Z, \mathcal{X})} \mathrm{Maps}(Z_2, \mathcal{X})$$

*is an isomorphism.*

*Proof.* Consider the following two inverse families of objects of  $\mathrm{DGSch}_{\mathrm{qc-qc}}$ :

$$n \mapsto {}^{L_{\mathcal{T}}\leq n}(Z') \text{ and } {}^{L_{\mathcal{T}}\leq n}(Z_1) \sqcup_{{}^{L_{\mathcal{T}}\leq n}(Z)} {}^{L_{\mathcal{T}}\leq n}(Z_2).$$

There is a natural map  $\leftarrow$ . The assumptions on the morphisms  $f_1$  and  $f_2$  imply that this map induces an isomorphism of  $m$ -coconnective truncations whenever  $m \ll n$ .

Therefore, for any  $\mathcal{X} \in \mathrm{DGindSch}$  (and indeed any  $\mathcal{X} \in {}^{\mathrm{conv}}\mathrm{PreStk}$ ), the induced map

$$\lim_n \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z'), \mathcal{X}) \rightarrow \lim_n \mathrm{Maps}\left({}^{L_{\mathcal{T}}\leq n}(Z_1) \sqcup_{{}^{L_{\mathcal{T}}\leq n}(Z)} {}^{L_{\mathcal{T}}\leq n}(Z_2), \mathcal{X}\right)$$

is an isomorphism.

Consider the composite map

$$\begin{aligned} & \mathrm{Maps}(Z', \mathcal{X}) \\ & \downarrow \\ & \mathrm{Maps}(Z_1, \mathcal{X}) \times_{\mathrm{Maps}(Z, \mathcal{X})} \mathrm{Maps}(Z_2, \mathcal{X}) \\ & \sim \downarrow \\ & \lim_n \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_1), \mathcal{X}) \times_{\lim_n \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z), \mathcal{X})} \lim_n \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_2), \mathcal{X}) \\ & \sim \downarrow \\ & \lim_n \left( \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_1), \mathcal{X}) \times_{\mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z), \mathcal{X})} \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_2), \mathcal{X}) \right). \end{aligned}$$

It equals the map

$$\begin{aligned} & \mathrm{Maps}(Z', \mathcal{X}) \\ & \sim \downarrow \\ & \lim_n \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z'), \mathcal{X}) \\ & \sim \downarrow \\ & \lim_n \mathrm{Maps}\left({}^{L_{\mathcal{T}}\leq n}(Z_1) \sqcup_{{}^{L_{\mathcal{T}}\leq n}(Z)} {}^{L_{\mathcal{T}}\leq n}(Z_2), \mathcal{X}\right) \\ & \downarrow \\ & \lim_n \left( \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_1), \mathcal{X}) \times_{\mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z), \mathcal{X})} \mathrm{Maps}({}^{L_{\mathcal{T}}\leq n}(Z_2), \mathcal{X}) \right), \end{aligned}$$

where the last arrow is an isomorphism by Proposition 3.4.1 above. This shows that

$$\mathrm{Maps}(Z', \mathcal{X}) \rightarrow \mathrm{Maps}(Z_1, \mathcal{X}) \times_{\mathrm{Maps}(Z, \mathcal{X})} \mathrm{Maps}(Z_2, \mathcal{X})$$

is an isomorphism as well. □

### 3.5. Presentation of indschemes.

3.5.1. First, we will prove point (a) of Proposition 1.4.3. We shall a slightly stronger (but, in fact, equivalent) statement; namely, we will prove Corollary 1.4.5.

*Proof.* We have to show that for  $Y \in \mathrm{DGSch}_{\mathrm{qc-qc}}^{\mathrm{aff}}$  and a map  $f : Y \rightarrow \mathcal{X}$ , the category of its factorizations

$$Y \rightarrow Z \rightarrow \mathcal{X},$$

where  $Z \in \mathrm{DGSch}_{\mathrm{qc-qc}}$ , and  $Z \rightarrow \mathcal{X}$  is a closed embedding is contractible.

By Corollary 3.2.5, the category in question admits push-outs and coproducts, and hence is filtered. Thus, to prove that it is contractible, it remains to show that it is non-empty.

Consider the map  ${}^{cl}f : {}^{cl}Y \rightarrow {}^{cl}\mathcal{X}$ . Since  ${}^{cl}\mathcal{X}$  is a classical indscheme, there exists a factorization

$${}^{cl}Y \xrightarrow{h_{cl}} Z_{cl} \xrightarrow{g_{cl}} {}^{cl}\mathcal{X},$$

where  $Z_{cl} \in \mathrm{Sch}_{\mathrm{qc-qc}}$  and  $g_{cl}$  is a closed embedding.

Let  $k$  be the cohomological amplitude of the functor  $(h_{cl})_* : \mathrm{QCoh}({}^{cl}Y) \rightarrow \mathrm{QCoh}(Z_{cl})$ , and let  $n$  be an integer  $> k$ .

Now, consider the truncation  $\leq^n Y$  and its map  $\leq^n f$  to  $\leq^n \mathcal{X}$ . Since  $\leq^n \mathcal{X}$  is a  $\leq^n \mathrm{DG}$  indscheme, the map  $\leq^n f$  can be factored as

$$\leq^n Y \xrightarrow{h_n} Z_n \xrightarrow{g_n} \leq^n \mathcal{X},$$

where  $Z_n \in \leq^n \mathrm{DGSch}_{\mathrm{qc-qc}}$  and  $g_n$  is a closed embedding. Moreover, without loss of generality, we can assume that we have a commutative square

$$\begin{array}{ccc} {}^{cl}(\leq^n Y) & \xrightarrow{{}^{cl}h_n} & {}^{cl}Z_n \\ \sim \uparrow & & \uparrow \\ {}^{cl}Y & \xrightarrow{h_{cl}} & Z_{cl}, \end{array}$$

where the right vertical map is automatically a closed embedding. In particular, we obtain that the cohomological amplitude of the functor  $({}^{cl}h_n)_*$  also equals  $k$ . Therefore, the same is true for the functor

$$(h_n)_* : \mathrm{QCoh}(\leq^n Y) \rightarrow \mathrm{QCoh}(Z_n).$$

(Note that in the above formula we are using the short-hand notation of Sect. 0.5.4 where instead of

$$L_T^{\leq^n}(Y) \simeq {}^L\mathrm{LKE}_{\leq^n \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\leq^n Y),$$

we simply write  $\leq^n Y$ , and instead of

$${}^L\mathrm{LKE}_{\leq^n \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(Z_n),$$

we write  $Z_n$ .)

Thus, the map  $h_n$  falls into the paradigm of Sect. 3.3.5. Now set

$$Z := Y \sqcup_{\leq^n Y} Z_n \in \mathrm{DGSch}.$$

By Corollary 3.4.2, we have a canonical map  $g : Z \rightarrow \mathcal{X}$ , which is a closed embedding since at the classical level this map is the same as  $g_n$ . Thus

$$Y \rightarrow Z \rightarrow \mathcal{X}$$

is the required factorization of  $f$ . □

3.5.2. Let us now prove Proposition 1.6.6. Our proof will rely on the notion of square-zero extension, which will be reviewed in Sect. 4.6.1.

We begin with the following observation:

**Lemma 3.5.3.** *Let  $\mathbf{C}$  be an  $\infty$ -category and  $i : \mathbf{C}_1 \rightarrow \mathbf{C}$  a fully faithful functor. Assume that  $\mathbf{C}$  is filtered. Then  $i$  is cofinal if and only if every object of  $\mathbf{C}$  admits a map to an object in  $\mathbf{C}_1$ . In this case  $\mathbf{C}_1$  is also filtered.*

We take  $\mathbf{C} := (\mathrm{DGSch}_{\mathrm{qc-qs}})_{\mathrm{closed}}$  in  $\mathcal{X}$  and  $\mathbf{C}_1 = (\mathrm{DGSch}_{\mathrm{aft}})_{\mathrm{closed}}$  in  $\mathcal{X}$ . Having proved Corollary 1.4.5, it remains to show that every closed embedding

$$f : Y \rightarrow X$$

admits a factorization

$$Y \rightarrow Z \xrightarrow{g} X,$$

where  $Z \in \mathrm{DGSch}_{\mathrm{aft}}$  and  $g$  is also a closed embedding.

*Step 1.* Consider a factorization of  ${}^{cl}f$

$${}^{cl}Y \xrightarrow{h_{cl}} Z_{cl} \xrightarrow{g_{cl}} {}^{cl}X,$$

where  $g_{cl}$  is a closed embedding. We claim that the “locally almost of finite type” assumption on  $\mathcal{X}$  implies that the classical scheme  $Z_{cl}$  is automatically of finite type.

This follows from the next lemma:

**Lemma 3.5.4.** *If  $\mathcal{X}_{cl}$  is a classical indscheme locally of finite type, and  $X_{cl} \rightarrow \mathcal{X}_{cl}$  a closed embedding, where  $X_{cl} \in \mathrm{Sch}$ , then  $X_{cl} \in \mathrm{Sch}_{\mathrm{ft}}$ .*

*Proof.* (of Lemma 3.5.4)

Note that a classical scheme  $X_{cl}$  is of finite type if and only if for any classical  $k$ -algebra  $A$  and a filtered family  $i \mapsto A_i$  of *subalgebras* such that  $A = \bigcup_i A_i$ , the map

$$\mathrm{colim}_i \mathrm{Maps}(\mathrm{Spec}(A_i), X_{cl}) \rightarrow \mathrm{Maps}(\mathrm{Spec}(A), X_{cl})$$

is an isomorphism.

Note that since  $A_i \rightarrow A$  are injective, the diagram

$$\begin{array}{ccc} \mathrm{colim}_i \mathrm{Maps}(\mathrm{Spec}(A_i), X_{cl}) & \longrightarrow & \mathrm{Maps}(\mathrm{Spec}(A), X_{cl}) \\ \downarrow & & \downarrow \\ \mathrm{colim}_i \mathrm{Maps}(\mathrm{Spec}(A_i), \mathcal{X}_{cl}) & \longrightarrow & \mathrm{Maps}(\mathrm{Spec}(A), \mathcal{X}_{cl}) \end{array}$$

is Cartesian. However, the bottom horizontal arrow is an isomorphism since  $\mathcal{X} \in {}^{cl}\mathrm{PreStk}_{\mathrm{ft}}$ .  $\square$

*Step 2.* We shall construct the required factorization of  $f$  by induction on  $n \geq 0$ . Namely, we shall construct a sequence of factorizations of  ${}^{\leq n}f : {}^{\leq n}Y \rightarrow {}^{\leq n}X$  as

$${}^{\leq n}Y \xrightarrow{h_n} Z_n \xrightarrow{g_n} {}^{\leq n}X,$$

with  $Z_n \in {}^{\leq n}\text{DGSch}_{\text{ft}}$ ,  $g_n$  a closed embedding, and such that for  $n \geq n'$ , we have a commutative diagram

$$\begin{array}{ccc} {}^{\leq n'}Z_n & \xrightarrow{{}^{\leq n'}g_n} & {}^{\leq n'}\mathcal{X} \\ \sim \uparrow & & \uparrow \text{id} \\ {}^{\leq n'}Z_{n'} & \xrightarrow{{}^{\leq n'}g_{n'}} & {}^{\leq n'}\mathcal{X}. \end{array}$$

Setting

$$Z := \text{colim}_n Z_n$$

(where the colimit is taken in  $\text{DGSch}$ ) we will then obtain the desired factorization of  $f$ .

*Step 3.* Suppose  $(Z_{n-1}, g_{n-1})$  have been constructed. Note that the maps

$$h_{n-1} : {}^{\leq n-1}Y \rightarrow Z_{n-1} \text{ and } {}^{\leq n-1}Y \rightarrow {}^{\leq n}Y$$

fall into the paradigm of Sect. 3.3.1. Set

$$Z'_n := Z_{n-1} \sqcup_{{}^{\leq n-1}Y} {}^{\leq n}Y.$$

We have  ${}^{\leq n-1}Z'_n \simeq {}^{\leq n-1}Z_{n-1}$ , and by Proposition 3.4.1 we obtain a natural map  $g'_n : Z'_n \rightarrow {}^{\leq n}\mathcal{X}$ .

To find the sought-for pair  $(Z_n, g_n)$ , it suffices to find a factorization of  $g'_n$  as

$$Z'_n \rightarrow Z_n \xrightarrow{g_n} {}^{\leq n}\mathcal{X},$$

so that  $Z_n \in {}^{\leq n}\text{DGSch}_{\text{ft}}$ , and  ${}^{\leq n-1}Z'_n \rightarrow {}^{\leq n-1}Z_n$  is an isomorphism.

*Step 4.* Note that the closed embedding

$${}^{\leq n-1}Y \rightarrow {}^{\leq n}Y$$

has a natural structure of a *square-zero extension*, see Sect. 4.6.1, by an ideal

$$\mathcal{J} \in \text{QCoh}({}^{\leq n-1}Y)^{\heartsuit}[n].$$

Hence, the closed embedding  $Z_{n-1} \rightarrow Z'_n$  also has a structure of a square-zero extension by

$$\mathcal{J} := (h_{n-1})_*(\mathcal{J}) \in \text{QCoh}(Z_{n-1})^{\heartsuit}[n].$$

*Step 5.* Write  $\mathcal{J}$  as a filtered colimit  $\text{colim}_{\alpha} \mathcal{J}_{\alpha}$ , where

$$\mathcal{J}_{\alpha} \in \text{Coh}(Z_{n-1})^{\heartsuit}[n].$$

The category  $\text{Coh}(Z_{n-1})$  is well-defined since  $Z_{n-1}$  is almost of finite type.

By Sect. 4.6.1, we obtain a family  $\alpha \mapsto Z_{n,\alpha}$  of objects of  ${}^{\leq n}\text{DGSch}$ , for all of which  ${}^{\leq n-1}Z_{n,\alpha} \simeq {}^{\leq n-1}Z_{n-1}$ ; moreover, we have isomorphisms

$$Z'_n \simeq \lim_{\alpha} Z_{n,\alpha}$$

as objects of  ${}^{\leq n}\text{DGSch}$ .

Now, since  $\mathcal{X}$  is locally almost of finite type as an object of  ${}^{\leq n}\text{PreStk}$ , the map

$$\text{colim}_{\alpha} \text{Maps}(Z_{n,\alpha}, {}^{\leq n}\mathcal{X}) \rightarrow \text{Maps}(Z'_n, {}^{\leq n}\mathcal{X})$$

is an isomorphism. In particular, the map  $g'_n$  factors through some  $g_{n,\alpha} : Z_{n,\alpha} \rightarrow {}^{\leq n}\mathcal{X}$ .

Now, the DG schemes  $Z_{n,\alpha}$  all belong to  $\leq^n \text{DGSch}_{\text{ft}}$ , by construction. This gives the required factorization.  $\square$

#### 4. DEFORMATION THEORY: RECOLLECTIONS

This section is preparation for Sect. 5. Our goal is the following: given  $\mathcal{X} \in \text{PreStk}$  such that  ${}^{cl}\mathcal{X}$  is a classical indscheme, we would like to give necessary and sufficient conditions for  $\mathcal{X}$  to be a DG indscheme. In this section we shall discuss what will be called Conditions (A), (B) and (C) that are satisfied by every DG indscheme. In Sect. 5 we will show that these conditions are also sufficient.

Conditions (A), (B) and (C) say that  $\mathcal{X}$  has a reasonable deformation theory. We will encode the property of being “reasonable” by the property of sending certain push-outs (in  $\text{DGSch}$ ) to Cartesian squares (in  $\infty\text{-Grpd}$ ).

##### 4.1. Split square-zero extensions and Condition (A).

4.1.1. *Split square-zero extensions.* For  $Z \in \leq^n \text{DGSch}$  and  $\mathcal{F} \in \text{QCoh}(Z)^{\geq -n, \leq 0}$  let  $Z_{\mathcal{F}}$  denote the corresponding split square-zero extension. I.e., locally in the Zariski topology if  $Z = S = \text{Spec}(A)$ , and  $\mathcal{M} := \Gamma(S, \mathcal{F})$ ,

$$S_{\mathcal{M}} := \text{Spec}(A \oplus \mathcal{M}),$$

where the multiplication on  $\mathcal{M}$  is zero.

Let  $\leq^n \text{SplitSqZExt}(Z)$  denote the category of split square-zero extensions, which by definition is equivalent to the category opposite to  $\text{QCoh}(Z)^{\geq -n, \leq 0}$ .

This category has push-outs: for  $\mathcal{F}_1, \mathcal{F}_2 \rightarrow \mathcal{F} \in \text{QCoh}(Z)^{\geq -n, \leq 0}$  the sought-for push-out is given by

$$\mathcal{F}' := \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2,$$

where the Cartesian product is taken in  $\text{QCoh}(Z)^{\geq -n, \leq 0}$ , i.e.,

$$\mathcal{F}' \simeq \tau^{\leq 0} \left( \mathcal{F}_1 \times_{\mathcal{F}} \mathcal{F}_2 \right).$$

By construction, the map

$$Z_{\mathcal{F}_1} \sqcup_{Z_{\mathcal{F}}} Z_{\mathcal{F}_2} \rightarrow Z_{\mathcal{F}'}$$

is an isomorphism, where the latter push-out is taken in the category  $\leq^n \text{DGSch}$  (i.e., when we forget the split square-zero extension structure). Moreover, if  $Z$  is affine, the above push-out agrees with the push-out in the category  $\leq^n \text{DGSch}^{\text{aff}}$ .

4.1.2. Let  $\mathcal{X}$  be an object of  $\leq^n \text{PreStk}$ . For  $S \in \leq^n \text{DGSch}^{\text{aff}}$  and a map  $x : S \rightarrow \mathcal{X}$ , consider the category  $\leq^n \text{SplitSqZExt}(S, x)$  consisting of triples

$$\{\mathcal{F} \in \text{QCoh}(S)^{\geq -n, \leq 0}, x' : S_{\mathcal{F}} \rightarrow \mathcal{X}, x'|_S \simeq x\}.$$

I.e.,

$$\leq^n \text{SplitSqZExt}(S, x) := \left( \leq^n \text{SplitSqZExt}(S) \times_{\leq^n \text{DGSch}} \leq^n \text{DGSch}_{/\mathcal{X}} \right) \times_{\text{Maps}(S, \mathcal{X})} x.$$

**Definition 4.1.3.** We shall say that  $\mathcal{X}$  satisfies indscheme-like Condition (A) if for any  $S$  and  $x$  as above, the category  $\leq^n \text{SplitSqZExt}(S, x)$  is filtered.



We can reformulate the above condition to a more familiar one. For an even more familiar formulation, the reader is invited to look at Sect. 4.2.1 below.

4.1.4. Consider the functor

$$\geq^{-n}(T_x^*\mathcal{X}) : \mathrm{QCoh}(S)^{\geq^{-n}, \leq^0} \rightarrow \infty\text{-Grpd}$$

defined by

$$(4.1) \quad \geq^{-n}(T_x^*\mathcal{X})(\mathcal{F}) := \{x' : S_{\mathcal{F}} \rightarrow \mathcal{X}, x'|_S \simeq x\}.$$

I.e.,

$$\mathcal{F} \mapsto \leq^n \mathrm{SplitSqZExt}(S, x) \times_{\leq^n \mathrm{SplitSqZExt}(S)} S_{\mathcal{F}}.$$

By [Lu0], Prop. 5.3.2.9, Condition (A) is equivalent to requiring that the functor  $\geq^{-n}(T_x^*\mathcal{X})$  send Cartesian products to Cartesian products.

4.1.5. As Cartesian products in  $\mathrm{QCoh}(S)^{\geq^{-n}, \leq^0}$  correspond to push-outs in  $\leq^n \mathrm{SplitSqZExt}(S)$ , we obtain that Condition (A) is equivalent to requiring that the functor

$$\leq^n \mathrm{SplitSqZExt}(S) \rightarrow \infty\text{-Grpd}$$

given by

$$(4.2) \quad (S \hookrightarrow S') \mapsto \{x' : S' \rightarrow \mathcal{X}, x'|_S \simeq x\} = \leq^n \mathrm{SplitSqZExt}(S, x) \times_{\leq^n \mathrm{SplitSqZExt}(S)} (S \hookrightarrow S')$$

take push-outs products to Cartesian products.

Since the forgetful functor

$$\leq^n \mathrm{SplitSqZExt}(S) \rightarrow \leq^n \mathrm{DGindSch}$$

sends push-outs to push-outs, from Proposition 3.4.1, we obtain that any  $\mathcal{X} \in \leq^n \mathrm{DGindSch}$  satisfies Condition (A).

4.1.6. Assume that  $\mathcal{X}$  satisfies Zariski descent. This allows us to extend  $\mathcal{X}$  to a functor

$$(\leq^n \mathrm{DGSch})^{op} \rightarrow \infty\text{-Grpd}$$

by

$$Z \mapsto \lim_{S \in \mathrm{Zar}(Z)} \mathrm{Maps}(S, \mathcal{X}),$$

where  $\mathrm{Zar}(Z)$  is the category of affine schemes endowed with an open embedding into  $X$ .

If  $\mathcal{X}$  satisfies Condition (A),  $Z \in \leq^n \mathrm{DGSch}$  and  $x : Z \rightarrow \mathcal{X}$  is a map, then the functor

$$\geq^{-n}(T_x^*\mathcal{X}) : \mathrm{QCoh}(Z)^{\geq^{-n}, \leq^0} \rightarrow \infty\text{-Grpd}$$

defined by the same formula as (4.1) also takes Cartesian products to Cartesian products.

## 4.2. Cotangent space as a pro-object.

4.2.1. Recall ([Lu0], Cor. 5.3.5.4) that for an arbitrary  $\infty$ -category  $\mathbf{C}$  that has Cartesian products, and a functor  $F : \mathbf{C} \rightarrow \infty\text{-Grpd}$ , the condition that  $F$  take Cartesian products to Cartesian products is equivalent to the condition that  $F$  be pro-representable.

Thus, we obtain that Condition (A) is equivalent to the condition that the functor  $\geq^{-n}(T_x^*\mathcal{X})$  be pro-representable.

Henceforth, whenever  $\mathcal{X}$  satisfies Condition (A), we shall denote by  $\geq^{-n}(T_x^*\mathcal{X})$  the corresponding object of  $\text{Pro}(\text{QCoh}(S)^{\geq^{-n}, \leq^0})$ . We shall refer to  $\geq^{-n}(T_x^*\mathcal{X})$  as “the cotangent space to  $\mathcal{X}$  at  $x : S \rightarrow \mathcal{X}$ ”.

Thus, an alternative terminology for Condition (A) is that the prestack  $\mathcal{X}$  *admits connective cotangent spaces*.<sup>14</sup>

Furthermore, note that in the situation of Sect. 4.1.6, the functor  $\geq^{-n}(T_x^*\mathcal{X})$  also corresponds to an object of  $\text{Pro}(\text{QCoh}(Z)^{\geq^{-n}, \leq^0})$ .

4.2.2. We shall now make a brief digression regarding the category  $\text{Pro}(\text{QCoh}(Z))$  for  $Z \in \text{DGSch}$ :

Let  $\mathbf{C}$  be a stable  $\infty$ -category. In this case, the category  $\text{Pro}(\mathbf{C})$  is also stable.<sup>15</sup> Assume now that  $\mathbf{C}$  is endowed with a t-structure. In this case  $\text{Pro}(\mathbf{C})$  also inherits a t-structure, so that its connective objects are those  $F \in \text{Pro}(\mathbf{C})$ , for which  $F(x) = 0$  for  $x \in \mathbf{C}^{>0}$ .

Restriction of functors defines a map

$$\text{Pro}(\mathbf{C})^{\leq 0} \rightarrow \text{Pro}(\mathbf{C}^{\leq 0}),$$

which is easily seen to be an equivalence. Similarly, for any  $n \geq 0$ , the natural functor

$$\text{Pro}(\mathbf{C})^{\geq^{-n}, \leq^0} \rightarrow \text{Pro}(\mathbf{C}^{\geq^{-n}, \leq^0})$$

is an equivalence.

Let  $\Phi : \mathbf{C}_1 \rightarrow \mathbf{C}_2$  be a functor between  $\infty$ -categories. Then the functor

$$\text{LKE}_\Phi : \text{Funct}(\mathbf{C}_1, \infty\text{-Grpd}) \rightarrow \text{Funct}(\mathbf{C}_2, \infty\text{-Grpd})$$

sends  $\text{Pro}(\mathbf{C}_1)$  to  $\text{Pro}(\mathbf{C}_2)$ ; we shall denote by

$$\text{Pro}(\Phi) : \text{Pro}(\mathbf{C}_1) \rightarrow \text{Pro}(\mathbf{C}_2)$$

the resulting functor.

If  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are stable and  $\Phi$  is exact, then so is  $\text{Pro}(\Phi)$ .

4.2.3. Now, assume that  $\mathbf{C}$  arises from a DG category (or, equivalently, is tensored over  $\text{Vect}$ ). Then any object  $F \in \text{Pro}(\mathbf{C})$ , viewed as a functor  $\mathbf{C} \rightarrow \infty\text{-Grpd}$ , canonically lifts to a functor

$$\mathbf{C} \rightarrow \text{Vect}^{\leq 0} \subset \text{Vect},$$

which by a slight abuse of notation we will denote by the same symbol  $F$ .

<sup>14</sup>We note that  $k$ -Artin stacks for  $k > 0$  viewed as objects of  $\text{PreStk}$  typically do not satisfy the above condition, as their cotangent spaces belong to  $\text{Pro}(\text{QCoh}(S)^{\geq^{-n}, \leq^k})$  but not to  $\text{Pro}(\text{QCoh}(S)^{\geq^{-n}, \leq^0})$ .

<sup>15</sup>Note, however, that even if  $\mathbf{C}$  is presentable, the category  $\text{Pro}(\mathbf{C})$  is not, so caution is required when applying such results as the adjoint functor theorem.

4.2.4. Let us now consider the following situation specific to  $\mathrm{QCoh}$ . Let  $Z$  be a DG scheme. Consider the following two categories

$$\mathrm{Pro}(\mathrm{QCoh}(Z)) \text{ and } \lim_{S \in \mathrm{Zar}(Z)} \mathrm{Pro}(\mathrm{QCoh}(S)).$$

Left Kan extension along

$$\mathcal{F} \mapsto \mathcal{F}|_S : \mathrm{QCoh}(Z) \rightarrow \mathrm{QCoh}(S)$$

defines a functor

$$(4.3) \quad \mathrm{Pro}(\mathrm{QCoh}(Z)) \rightarrow \lim_{S \in \mathrm{Zar}(Z)} \mathrm{Pro}(\mathrm{QCoh}(S)).$$

This functor admits a right adjoint, which is tautologically described as follows. To

$$\{S \mapsto (F_S \in \mathrm{Pro}(\mathrm{QCoh}(S)))\} \in \lim_{S \in \mathrm{Zar}(Z)} \mathrm{Pro}(\mathrm{QCoh}(S))$$

it assigns  $F \in \mathrm{Pro}(\mathrm{QCoh}(Z))$  defined by

$$F(\mathcal{F}) := \lim_{S \in \mathrm{Zar}(Z)} F_S(\mathcal{F}|_S).$$

We claim:

**Lemma 4.2.5.** *Assume that  $Z$  is quasi-compact and quasi-separated. Then the above two functors*

$$(4.4) \quad \mathrm{Pro}(\mathrm{QCoh}(Z)) \rightleftarrows \lim_{S \in \mathrm{Zar}(Z)} \mathrm{Pro}(\mathrm{QCoh}(S))$$

*are mutually inverse.*

*Proof.* A standard argument shows that instead of  $\mathrm{Zar}(S)$  we can consider a *finite* limit corresponding to a Zariski hypercovering.

Note that the left Kan extension  $\mathrm{Pro}(\mathrm{QCoh}(Z)) \rightarrow \mathrm{Pro}(\mathrm{QCoh}(S))$  can be also expressed as the functor

$$F \mapsto F \circ (j_S)_*,$$

where  $j_S$  denotes the open embedding  $S \hookrightarrow Z$ .

Then the fact that the two adjunction maps are isomorphisms follows from the fact that

$$\mathrm{Id}_{\mathrm{QCoh}(S)} \rightarrow \lim_S (j_S)_* \circ j_S^*$$

is an isomorphism and the functors  $F$  and  $F_S$  involved commute with finite limits. □

Note that the lemma (with the same proof) also applies when we replace the category  $\mathrm{Pro}(\mathrm{QCoh}(Z))$  by  $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n, \leq 0})$  for any  $n \geq 0$ .

### 4.3. The relative situation.

4.3.1. For future reference, let us note that the functor  $\geq^{-n}(T_x^*\mathcal{X})$  can be defined in a relative situation, i.e., when we are dealing with a map of prestacks  $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ . Namely, for  $x : S \rightarrow \mathcal{X}$  as above, we set  $T_x^*\mathcal{X}/\mathcal{Y}$  to be the functor

$$\mathrm{QCoh}(S)^{\geq -n, \leq 0} \rightarrow \infty\text{-Grpd}$$

defined by

$$\mathcal{F} \mapsto \leq^n \mathrm{SplitSqZExt}(S, x) \times_{\leq^n \mathrm{SplitSqZExt}(S, \phi \circ x)} S_{\mathcal{F}},$$

where  $S_{\mathcal{F}}$  defines the point of  $\leq^n \mathrm{SplitSqZExt}(S, \phi \circ x)$  equal to the composition

$$S_{\mathcal{F}} \xrightarrow{\pi} S \xrightarrow{\phi \circ x} \mathcal{Y},$$

and where  $\pi : S_{\mathcal{F}} \rightarrow S$  is the canonical projection.

4.3.2. Note that if both  $\mathcal{X}$  and  $\mathcal{Y}$  admit connective cotangent spaces,  $T_x^*\mathcal{X}/\mathcal{Y}$  identifies with the object of  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$  given by

$$\tau^{\geq -n}(\mathrm{Cone}(T_{\phi \circ x}^* \mathcal{Y} \rightarrow T_x^* \mathcal{X})).$$

#### 4.4. Functoriality of split square-zero extensions and Condition (B).

4.4.1. Let  $\phi : Z_1 \rightarrow Z$  be a map between objects of  $\leq^n \mathrm{DGSch}$ . Direct image  $\phi_*$  composed with the truncation  $\tau^{\leq 0}$  defines a functor

$$\leq^0 \phi_* : \mathrm{QCoh}(Z_1)^{\geq -n, \leq 0} \rightarrow \mathrm{QCoh}(Z)^{\geq -n, \leq 0},$$

which corresponds to the push-out functor

$$\leq^n \mathrm{SplitSqZExt}(Z_1) \rightarrow \leq^n \mathrm{SplitSqZExt}(Z) : Z'_1 \mapsto Z \sqcup_{Z_1} Z'_1.$$

4.4.2. Assume now that  $Z = S$  and  $Z_1 = S_1$  are affine. Let  $\mathcal{X}$  be an object of  $\leq^n \mathrm{PreStk}$ , and  $x$  an  $S$ -point of  $\mathcal{X}$ . Set  $x_1 := x \circ \phi : S_1 \rightarrow \mathcal{X}$ . Composition defines a functor

$$(4.5) \quad \leq^n \mathrm{SplitSqZExt}(S_1) \times_{\leq^n \mathrm{SplitSqZExt}(S)} \leq^n \mathrm{SplitSqZExt}(S, x) \rightarrow \leq^n \mathrm{SplitSqZExt}(S_1, x_1).$$

**Definition 4.4.3.** *We shall say that  $\mathcal{X} \in \leq^n \mathrm{PreStk}$  satisfies indscheme-like Condition (B) if the above functor is an equivalence for any  $(S, S_1, \phi)$ .*

4.4.4. We can tautologically rephrase Condition (B) as follows: for  $\mathcal{F}_1 \in \mathrm{QCoh}(S_1)^{\geq -n, \leq 0}$  and  $\mathcal{F} := \phi_*(\mathcal{F}_1) \in \mathrm{QCoh}(S)^{\geq -n, \leq 0}$ , the map

$$\{x' : S_{\mathcal{F}} \rightarrow \mathcal{X}, x'|_S \simeq x\} \rightarrow \{x'_1 : (S_1)_{\mathcal{F}_1} \rightarrow \mathcal{X}, x'_1|_{S_1} \simeq x_1\},$$

i.e.,

$$\leq^n \mathrm{SplitSqZExt}(S, x) \times_{\leq^n \mathrm{SplitSqZExt}(S)} \mathcal{F} \rightarrow \leq^n \mathrm{SplitSqZExt}(S_1, x_1) \times_{\leq^n \mathrm{SplitSqZExt}(S_1)} \mathcal{F}_1$$

is an isomorphism in  $\infty\text{-Grpd}$ .

4.4.5. Another tautological reformulation of Condition (B) is the following: the presheaf  $\mathcal{X}$  takes push-outs in  $\leq^n \mathrm{DGSch}$  of the form  $S \sqcup_{S_1} (S_1)_{\mathcal{F}_1}$  to Cartesian squares.

This implies that if  $\mathcal{X}$  belongs to  $\leq^n \mathrm{DGindSch}$ , then it automatically satisfies Condition (B).

4.4.6. Condition (B) can also be tautologically rephrased as follows: consider the functor

$$\geq^{-n}\phi^* : \mathrm{QCoh}(S)^{\geq -n, \leq 0} \rightarrow \mathrm{QCoh}(S_1)^{\geq -n, \leq 0}$$

given by  $\mathcal{F} \mapsto \tau^{\geq -n}(\phi^*(\mathcal{F}))$ . Condition (B) is equivalent to the requirement that the natural map

$$\mathrm{LKE}_{\geq -n\phi^*}(\geq^{-n}(T_x^*\mathcal{X})) \rightarrow \geq^{-n}(T_{x_1}^*\mathcal{X}).$$

be an isomorphism, where  $\geq^{-n}(T_x^*\mathcal{X})$  and  $\geq^{-n}(T_{x_1}^*\mathcal{X})$  are viewed as functors

$$\mathrm{QCoh}(S)^{\geq -n, \leq 0} \rightarrow \infty\text{-Grpd} \text{ and } \mathrm{QCoh}(S_1)^{\geq -n, \leq 0} \rightarrow \infty\text{-Grpd},$$

respectively.

4.4.7. Now, suppose that  $\mathcal{X}$  satisfies Condition (A). The functor

$$\mathrm{LKE}_{\geq -n\phi^*} : \mathrm{Func}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}, \infty\text{-Grpd}) \rightarrow \mathrm{Func}(\mathrm{QCoh}(S_1)^{\geq -n, \leq 0}, \infty\text{-Grpd}),$$

restricted to

$$\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}) \subset \mathrm{Func}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}, \infty\text{-Grpd}),$$

identifies with the functor

$$\mathrm{Pro}(\geq^{-n}\phi^*) : \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}) \rightarrow \mathrm{Pro}(\mathrm{QCoh}(S_1)^{\geq -n, \leq 0}).$$

Thus, for a map  $\phi : S_1 \rightarrow S$  as above, we obtain a natural map

$$(4.6) \quad \geq^{-n}(T_{x_1}^*\mathcal{X}) \rightarrow \mathrm{Pro}(\geq^{-n}\phi^*)(\geq^{-n}(T_x^*\mathcal{X})),$$

where we now view  $\geq^{-n}(T_x^*\mathcal{X})$  and  $\geq^{-n}(T_{x_1}^*\mathcal{X})$  as objects of

$$\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}) \text{ and } \mathrm{Pro}(\mathrm{QCoh}(S_1)^{\geq -n, \leq 0}),$$

respectively.

In this case, Condition (B) can be reformulated as the requirement that the map (4.6) be an isomorphism.

**Definition 4.4.8.** *We shall say that  $\mathcal{X} \in {}^{\leq n}\mathrm{PreStk}$  admits a connective cotangent complex if it satisfies both Conditions (A) and (B).*

In other words,  $\mathcal{X}$  admits a connective cotangent complex if it admits connective cotangent spaces, whose formation is compatible with pullbacks under morphisms of affine DG schemes.

4.4.9. Let us now assume that  $\mathcal{X}$  satisfies Zariski descent, as well as Conditions (A) and (B).

Thus, for  $Z \in {}^{\leq n}\mathrm{DGSch}$  and  $x : Z \rightarrow \mathcal{X}$ , we have a well-defined object

$$\geq^{-n}(T_x^*\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n, \leq 0}).$$

We wish to compare the restriction of  $\geq^{-n}(T_x^*\mathcal{X})$  to a given affine Zariski open  $S \subset Z$  with

$$\geq^{-n}(T_{x|_S}^*\mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0}).$$

As in (4.6), we have a natural map

$$(4.7) \quad \geq^{-n}(T_{x|_S}^*\mathcal{X}) \rightarrow \geq^{-n}(T_x^*\mathcal{X})|_S.$$

We claim:

**Lemma 4.4.10.** *Assume that  $Z$  is quasi-compact and quasi-separated. Then the map (4.7) is an isomorphism.*

*Proof.* This follows from the description of  $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n, \leq 0})$  given by Lemma 4.2.5.  $\square$

#### 4.5. Cotangent complex of a DG scheme.

4.5.1. Assume for a moment that  $\mathcal{X} = X \in {}^{\leq n}\text{DGSch}$ . It is well-known that in this case the object  ${}^{\geq -n}(T_x^*X) \in \text{Pro}(\text{QCoh}(S)^{\geq -n, \leq 0})$  actually belongs to  $\text{QCoh}(S)^{\geq -n, \leq 0}$ :

*Proof.* It is enough to show that the functor  ${}^{\geq -n}(T_x^*X)$  commutes with filtered limits. But filtered limits in  $\text{QCoh}(S)^{\geq -n, \leq 0}$  map to filtered colimits in  ${}^{\leq n}\text{DGSch}^{\text{aff}}$ , and the assertion follows tautologically.  $\square$

4.5.2. We obtain that for any  $X \in {}^{\leq n}\text{DGSch}$  we have a well-defined object  ${}^{\geq -n}(T^*X) \in \text{QCoh}(X)^{\geq -n, \leq 0}$ , such that for any affine  $S$  with a map  $x : S \rightarrow X$ , we have

$$(4.8) \quad {}^{\geq -n}(T_x^*X) \simeq {}^{\geq -n}x^*({}^{\geq -n}(T^*X)).$$

Moreover, as schemes are sheaves in the Zariski topology, the isomorphism (4.8) remains valid when  $S \in {}^{\leq n}\text{DGSch}^{\text{aff}}$  is replaced by an arbitrary object  $Z \in {}^{\leq n}\text{DGSch}$ .

4.5.3. In particular, taking  $Z = X$  and  $x$  to be the identity map, we obtain that the identity map on  ${}^{\geq -n}(T^*Z)$  defines a canonical map

$$\mathfrak{d}_{can} : Z_{\geq -n}(T^*Z) \rightarrow Z.$$

4.5.4. Assume now that  $\mathcal{X} \in {}^{\leq n}\text{DGindSch}$ , and is written as in (1.1) for some index set  $A$ , and let  $Z \in {}^{\leq n}\text{DGSch}_{\text{qc-qs}}$ .

Let  $x : Z \rightarrow \mathcal{X}$  be a map that factors through a map  $x_{\alpha_0} : Z \rightarrow X_{\alpha_0}$ . We obtain that  ${}^{\geq -n}(T_x^*\mathcal{X})$  can be explicitly presented as a pro-object of  $\text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$ . Namely, we have:

$${}^{\geq -n}(T_x^*\mathcal{X}) \simeq \varprojlim_{\alpha \in A_{\alpha_0/}} {}^{\geq -n}(T_{x_\alpha}^*X_\alpha),$$

where  $x_\alpha$  denotes the composition  $Z \xrightarrow{x_{\alpha_0}} X_{\alpha_0} \rightarrow X_\alpha$ .

4.5.5. Let  $\mathcal{X}$  again be an arbitrary object of  ${}^{\leq n}\text{PreStk}$ , satisfying Condition (A),  $S \in {}^{\leq n}\text{DGSch}^{\text{aff}}$  and  $x : S \rightarrow \mathcal{X}$  a point. We claim that there exists a canonical map in  $\text{Pro}(\text{QCoh}(S)^{\geq -n, \leq 0})$

$$(4.9) \quad (dx)^* : {}^{\geq -n}(T_x^*\mathcal{X}) \rightarrow {}^{\geq -n}(T^*S).$$

Indeed, it corresponds to the map  $S_{\geq -n}(T^*S) \rightarrow \mathcal{X}$  equal to the composition

$$S_{\geq -n}(T^*S) \xrightarrow{\mathfrak{d}_{can}} S \xrightarrow{x} \mathcal{X}.$$

The same remains true with  $S \in {}^{\leq n}\text{DGSch}^{\text{aff}}$  replaced by  $Z \in {}^{\leq n}\text{DGSch}$ , whenever  $\mathcal{X}$  satisfies Zariski descent.

#### 4.6. General square-zero extensions.

4.6.1. Let  $Z$  and  $Z'$  be objects of  ${}^{\leq n-1}\text{DGSch}$ , and let  $Z \hookrightarrow Z'$  be a closed embedding given by a square-zero ideal.

This ideal can be thought of as an object  $\mathcal{I} \in \text{QCoh}(Z)^{\geq -n+1, \leq 0}$ , and the datum of  $Z'$  is equivalent to the datum of a map

$${}^{\geq -n}(T^*Z) \rightarrow \mathcal{I}[1]$$

in  $\text{QCoh}(Z)$ .

4.6.2. Explicitly, for a map  $\gamma : \geq^{-n}(T^*Z) \rightarrow \mathcal{J}[1]$  we construct  $Z'$  as the push-out in  $\leq^n \text{DGSch}$

$$(4.10) \quad Z \sqcup_{Z_{\mathcal{J}[1]}} Z,$$

where one of the maps  $Z_{\mathcal{J}[1]} \rightarrow Z$  is the projection, and the other corresponds to  $\gamma$  via the universal property of  $\geq^{-n}(T^*Z)$ .<sup>16</sup>

4.6.3. We shall denote the category of square-zero extensions by  $\leq^{n-1} \text{SqZExt}(Z)$ . By the above, this category is equivalent to

$$((\text{QCoh}(Z))^{\geq -n+1, \leq 0})_{\geq^{-n}(T^*Z)[-1]/}^{op}$$

of objects of  $\text{QCoh}(Z)^{\geq -n+1, \leq 0}$  under  $\geq^{-n}(T^*Z)[-1] \in \text{QCoh}(Z)^{\geq -n+1, \leq 1}$ .

*Remark.* Let us emphasize that, unlike the situation of classical schemes, the forgetful functor

$$\leq^{n-1} \text{SqZExt}(Z) \rightarrow \leq^{n-1} \text{DGSch}_{Z/}$$

is *not* fully faithful. I.e., being a square-zero extension is not a property, but is extra structure.

4.6.4. Let  $\phi : Z_1 \rightarrow Z$  be an affine map between objects of  $\leq^{n-1} \text{DGSch}$ . Push-out defines a functor

$$(4.11) \quad \leq^{n-1} \text{SqZExt}(Z_1) \rightarrow \leq^{n-1} \text{SqZExt}(Z) : Z'_1 \mapsto Z' := Z'_1 \sqcup_{Z_1} Z.$$

4.6.5. We can equivalently describe this construction as follows. Suppose that  $Z'_1$  corresponds to  $\mathcal{J}_1 \in \text{QCoh}(Z_1)^{\geq -n+1, \leq 0}$  and a map  $\gamma_1 : \geq^{-n}(T^*Z_1) \rightarrow \mathcal{J}[1]$ . Composing with

$$(d\phi)^* : \geq^{-n}\phi^*(\geq^{-n}(T^*Z)) \rightarrow \geq^{-n}(T^*Z_1),$$

we obtain a map

$$\geq^{-n}\phi^*(\geq^{-n}(T^*Z)) \rightarrow \mathcal{J}_1[1],$$

and by adjunction a map

$$(4.12) \quad \gamma : \geq^{-n}(T^*Z) \rightarrow \phi_*(\mathcal{J}_1)[1].$$

Setting  $\mathcal{J} := \phi_*(\mathcal{J}_1)$  we thus obtain a square-zero extension  $Z'$  of  $Z$ . It is easy to see that it is, in fact, canonically isomorphic to the push-out above. In fact, the map (4.12) can be interpreted as a map  $Z_{\mathcal{J}[1]} \rightarrow Z$  equal to

$$Z_{\mathcal{J}[1]} \simeq Z \sqcup_{Z_1} (Z_1)_{\mathcal{J}_1[1]} \xrightarrow{\text{id}_Z \sqcup \gamma_1} Z \sqcup_{Z_1} Z_1 \simeq Z,$$

and we have an isomorphism

$$(4.13) \quad (Z_1 \sqcup_{(Z_1)_{\mathcal{J}_1[1]}} Z_1) \sqcup_{Z_1} Z \simeq Z \sqcup_{Z_1 \sqcup_{(Z_1)_{\mathcal{J}_1[1]}}} Z.$$

#### 4.7. Infinitesimal cohesiveness and Condition (C).

<sup>16</sup>Here we are using the notational convention of Sect. 0.5.4 by denoting by  $Z$  the object of  $\leq^n \text{DGSch}$  that should properly be denoted  $\leq^n L_{\text{LKE} \leq^{n-1} \text{DGSch} \hookrightarrow \leq^n \text{DGSch}}(Z)$ .

4.7.1. Let  $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$  and  $\mathcal{I} \in \mathrm{QCoh}(S)^{\geq -n+1, \leq 0}$ , and let  $\gamma : T^*S \rightarrow \mathcal{I}[1]$  be a map. Let  $S' \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$  be the corresponding square-zero extension given by (4.10).

For  $\mathcal{X} \in {}^{\leq n}\mathrm{PreStk}$ , regarding  $S_{\mathcal{I}[1]}$  as an affine  ${}^{\leq n}\mathrm{DG}$  scheme, we obtain a map

$$(4.14) \quad \mathrm{Maps}(S', \mathcal{X}) \rightarrow \mathrm{Maps}(S, \mathcal{X}) \times_{\mathrm{Maps}(S_{\mathcal{I}[1]}, \mathcal{X})} \mathrm{Maps}(S, \mathcal{X}).$$

**Definition 4.7.2.** We shall say that  $\mathcal{X}$  satisfies *indscheme-like Condition (C)* if (4.14) is an isomorphism for any  $(S, \mathcal{I}, \gamma)$  as above.

An alternative terminology for prestacks satisfying Condition (C) is *infinitesimally cohesive*.

4.7.3. If  $\mathcal{X}$  belongs to  ${}^{\leq n}\mathrm{DGindSch}$ , it satisfies Condition (C) by Proposition 3.4.1.

4.7.4. For  $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$ , let  ${}^{\leq n-1}\mathrm{SqZExt}(S, x)$  be the category of triples

$$\{S \hookrightarrow S', x' : S' \rightarrow \mathcal{X}, x'|_S \simeq x\},$$

where  $S \hookrightarrow S'$  is a square-zero extension with  $S' \in {}^{\leq n-1}\mathrm{DGSch}$ . I.e.,

$${}^{\leq n-1}\mathrm{SqZExt}(S, x) := \left( {}^{\leq n-1}\mathrm{SqZExt}(S) \times_{{}^{\leq n}\mathrm{DGSch}} {}^{\leq n-1}\mathrm{DGSch}_{/\mathcal{X}} \right) \times_{\mathrm{Maps}(S, \mathcal{X})} x.$$

Suppose now that  $\mathcal{X}$  satisfies Condition (A). For  $S \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$ , recall the map in  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$

$$(dx)^* : {}^{\geq -n}(T_x^*\mathcal{X}) \rightarrow {}^{\geq -n}(T^*S).$$

Consider the object

$$\mathrm{Cone}((dx)^*)[-1] \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n+1, \leq 1}).$$

We can reformulate Condition (C) as follows: the natural functor

$${}^{\leq n-1}\mathrm{SqZExt}(S, x) \rightarrow \left( (\mathrm{QCoh}(S)^{\geq -n+1, \leq 0})_{\mathrm{Cone}((dx)^*)[-1]/} \right)^{op}$$

is an equivalence, where the right-hand side is the opposite of the category of objects of  $\mathrm{QCoh}(S)^{\geq -n+1, \leq 0}$  under  $\mathrm{Cone}((dx)^*)[-1]$ .

4.7.5. Now, assume that  $\mathcal{X}$  satisfies Zariski descent as well as Conditions (A) and (C). We obtain that for  $Z \in {}^{\leq n-1}\mathrm{DGSch}^{\mathrm{aff}}$  and a given  $x : Z \rightarrow \mathcal{X}$ , the description of the category  ${}^{\leq n-1}\mathrm{SqZExt}(Z, x)$  as

$$\left( (\mathrm{QCoh}(Z)^{\geq -n+1, \leq 0})_{\mathrm{Cone}((dx)^*)[-1]/} \right)^{op}$$

remains valid.

Moreover, we have the following:

**Lemma 4.7.6.** *Under the above circumstances the following are equivalent:*

- (a) *The category  ${}^{\leq n-1}\mathrm{SqZExt}(Z, x)$  is filtered.*
- (b)  *$\mathrm{Cone}((dx)^*)[-1]$  belongs to  $\mathrm{Pro}(\mathrm{QCoh}(Z)^{\geq -n+1, \leq 0})$ .*
- (c) *The map*

$$H^0((dx)^*) : H^0({}^{\geq -n}(T_x^*\mathcal{X})) \rightarrow H^0({}^{\geq -n}(T^*Z))$$

*is surjective.*



4.7.7. Let  $\phi : S_1 \rightarrow S$  be a map in  $\leq^{n-1}\text{DGSch}^{\text{aff}}$ . Recall the functor (4.11). For  $x : S \rightarrow \mathcal{X}$ , composition defines a map

$$(4.15) \quad \leq^{n-1}\text{SqZExt}(S, x) \times_{\leq^{n-1}\text{SqZExt}(S)} \leq^{n-1}\text{SqZExt}(S_1) \rightarrow \leq^{n-1}\text{SqZExt}(S_1, x_1).$$

The discussion in Sect. 4.6.5 implies:

**Lemma 4.7.8.** *If  $\mathcal{X}$  satisfies Conditions (B) and (C), then the map (4.15) is an isomorphism.*

If  $\mathcal{X}$  satisfies Zariski descent, then the same continues to be true for  $S$  and  $S_1$  replaced by arbitrary objects  $Z, Z_1 \in \leq^{n-1}\text{DGSch}$ , but keeping the requirement that the map  $Z_1 \rightarrow Z$  be affine. We can rewrite this isomorphism as follows:

For a given  $(Z_1 \hookrightarrow Z'_1) \in \leq^{n-1}\text{SqZExt}(Z_1)$  and  $x : Z \rightarrow \mathcal{X}$ , the map

$$(4.16) \quad \begin{aligned} & \text{colim}_{(Z \hookrightarrow Z', x') \in \leq^{n-1}\text{SqZExt}(Z, x)} \left( (Z \hookrightarrow Z') \times_{\leq^{n-1}\text{SqZExt}(Z)} (Z_1 \hookrightarrow Z'_1) \right) \rightarrow \\ & \rightarrow \leq^{n-1}\text{SqZExt}(Z_1, x_1) \times_{\leq^{n-1}\text{SqZExt}(Z_1)} (Z_1 \hookrightarrow Z'_1) \end{aligned}$$

is an isomorphism in  $\infty\text{-Grpd}$ .

**4.8. Dropping  $n$ -coconnectivity.** Finally, we note that the above considerations are valid for an object  $\mathcal{X} \in \text{PreStk}$ , simply by omitting the  $n$ -coconnectivity conditions.

**Definition 4.8.1.** *We shall say that  $\mathcal{X} \in \text{PreStk}$  admits connective deformation theory if it is convergent, and satisfies indscheme-like Conditions (A), (B) and (C).*

## 5. A CHARACTERIZATION OF DG INDSCHEMES VIA DEFORMATION THEORY

**5.1. The statement.** Let  $\mathcal{X}$  be an object of  $\leq^n \text{PreStk}$ , such that  ${}^{cl}\mathcal{X}$  is a classical indscheme. We would like to give a criterion for when  $\mathcal{X}$  belongs to  $\leq^n \text{DGindSch}$ .

**Theorem 5.1.1.** *Under the above circumstances,  $\mathcal{X} \in \leq^n \text{DGindSch}$  if and only if  $\mathcal{X}$  admits an extension to an object  $\mathcal{X}_{n+1} \in \leq^{n+1} \text{PreStk}$ , which satisfies indscheme-like Conditions (A), (B) and (C).*

The rest of this subsection is devoted to the proof of this theorem. The “only if” direction is clear: if  $\mathcal{X} \in \leq^n \text{DGindSch}$ , the extension

$$\mathcal{X}_{n+1} := \leq^{n+1} L\text{LKE}_{\leq^n \text{DGSch}^{\text{aff}} \hookrightarrow \leq^{n+1} \text{DGSch}^{\text{aff}}}(\mathcal{X})$$

belongs to  $\leq^{n+1} \text{DGindSch}$ , and hence satisfies Conditions (A), (B) and (C).

For the opposite implication, we will argue by induction on  $n$ , assuming that the statement is true for  $n' < n$ . In particular, we can assume that  $\leq^{n-1} \mathcal{X} := \mathcal{X}|_{\leq^{n-1} \text{DGSch}^{\text{aff}}}$  belongs to  $\leq^{n-1} \text{DGindSch}$ .

### 5.1.2. Step 0: initial remarks.

First, note that the induction hypothesis combined with Condition (C) implies that the prestack  $\mathcal{X}$  satisfies Zariski descent. Hence, deformation theory of maps into it from objects of  $\leq^n \text{DGSch}_{\text{qc-qs}}$ , described in the previous section, is applicable.

Thus, for  $X \in \leq^{n-1} \text{DGSch}$  and a map  $f : X \rightarrow \leq^{n-1} \mathcal{X}$ , we have a well-defined object

$$\geq^{-n-1}(T_f^* \mathcal{X}_{n+1}) \in \text{Pro}(\text{QCoh}(X)^{\geq -n-1, \leq 0}),$$

the formation of which is compatible with pull-backs.

Moreover, when  $X$  is quasi-compact and quasi-separated, we have:

$$\leq^n \text{SplitSqZExt}(X, f) \simeq \left( (\text{QCoh}(X)^{\geq -n, \leq 0})_{\geq -n-1(T_f^* \mathcal{X}_{n+1})} \right)^{op}.$$

Let  $\leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}$  denote the full subcategory of  $\leq^{n-1} \text{DGSch}_{/\mathcal{X}}$  that consists of  $(X, f) \in \leq^{n-1} \text{DGSch}_{/\mathcal{X}}$  for which  $f$  is a closed embedding. In particular, any such  $X$  is quasi-compact and quasi-separated, and for any map

$$(X_1, f_1) \rightarrow (X, f)$$

in this category given by

$$(\phi : X_1 \rightarrow X, f_1 \simeq \phi \circ f),$$

where the underlying map  $\phi : X_1 \rightarrow X$  is also a closed embedding, and in particular, affine.

We obtain that push-out makes the assignment

$$(X, f) \mapsto \leq^n \text{SqZExt}(X, f)$$

into a category co-fibered over  $\leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}$ . We denote it by

$$\leq^n \text{SqZExt}(\leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}).$$

Also, note that Lemma 4.7.6 implies that for any  $(X, f) \in \leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}$ , the category  $\leq^n \text{SqZExt}(X, f)$  is filtered.

### 5.1.3. Step 1: creating closed embeddings.

It is of course not true that for any  $(X, f) \in \leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}$  and

$$(i : X \hookrightarrow X', f' : X' \rightarrow \mathcal{X}) \in \leq^n \text{SqZExt}(X, f),$$

the map  $f'$  is also a closed embedding.

Let

$$\leq^n \text{SqZExt}(X, f)_{\text{closed in } \mathcal{X}} \subset \leq^n \text{SqZExt}(X, f)$$

denote the full subcategory spanned by objects for which the map  $f'$  is a closed embedding.

From Sect. 3.2.7, we obtain that the embedding

$$(5.1) \quad \leq^n \text{SqZExt}(X, f)_{\text{closed in } \mathcal{X}} \hookrightarrow \leq^n \text{SqZExt}(X, f)$$

admits a left adjoint. In particular, the above embedding is cofinal.

Let

$$\leq^n \text{SqZExt}(\leq^{n-1} \text{DGSch}_{/\mathcal{X}})_{\text{closed in } \mathcal{X}}$$

denote the corresponding full subcategory of  $\leq^n \text{SqZExt}(\leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}})$ . We claim that the forgetful functor

$$\leq^n \text{SqZExt}(\leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}) \rightarrow \leq^{n-1} \text{DGSch}_{\text{closed in } \mathcal{X}}$$

is still a co-Cartesian fibration. Indeed, for a map  $\phi : X_1 \rightarrow X$  in  $\leq^{n-1}\text{DGSch}/_{\text{closed in } \mathcal{X}}$  the corresponding map

$$\leq^n \text{SqZExt}(X_1, f_1)_{\text{closed in } \mathcal{X}} \rightarrow \leq^n \text{SqZExt}(X, f)_{\text{closed in } \mathcal{X}}$$

is given by composing

$$\leq^n \text{SqZExt}(X_1, f_1)_{\text{closed in } \mathcal{X}} \hookrightarrow \leq^n \text{SqZExt}(X_1, f_1) \rightarrow \leq^n \text{SqZExt}(X, f)$$

with the above left adjoint to (5.1).

5.1.4. *Step 2: construction of the inductive system.* Let

$$\leq^{n-1}\mathcal{X} \simeq \text{colim}_{\alpha \in A} X_\alpha$$

be a presentation as in (1.1) with  $X_\alpha \in \leq^{n-1}\text{DGSch}_{\text{qc-qs}}$ . For every  $\alpha \in A$ , let  $f_\alpha$  denote the corresponding map  $X_\alpha \rightarrow \leq^{n-1}\mathcal{X}$ . For an arrow  $\alpha_1 \rightarrow \alpha_2$ , let  $f_{\alpha_1, \alpha_2}$  denote the corresponding map  $X_{\alpha_1} \rightarrow X_{\alpha_2}$ .

For each  $\alpha$ , let  $\mathbf{B}_\alpha$  denote the category

$$\leq^n \text{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in } \mathcal{X}}.$$

Denoting by  $\beta$  an object of  $\mathbf{B}_\alpha$ , we will denote by  $X_\beta$  the corresponding  $\leq^n \text{DG}$  scheme  $X'_\alpha$ , and by  $f_\beta$  the closed embedding  $f'_\alpha$ . Let  $i_\beta$  denote the closed embedding  $X_\alpha \rightarrow X_\beta$ .<sup>17</sup> We have an evident functor from  $\mathbf{B}_\alpha$  to the category of  $\leq^n \text{DG}$  schemes endowed with a closed embedding into  $\mathcal{X}$ .

The above construction makes the assignment

$$\alpha \mapsto \mathbf{B}_\alpha$$

into a category co-fibered over  $A$ . Let  $\phi$  denote the tautological map  $\mathbf{B} \rightarrow A$ . Since  $A$  is filtered and all  $\mathbf{B}_\alpha$  are filtered, the category  $\mathbf{B}$  is also filtered.

It is also clear that the assignment

$$(\beta \in \mathbf{B}) \mapsto (X_\beta \xrightarrow{f_\beta} \mathcal{X})$$

is a functor from  $\mathbf{B}$  to the category of  $\leq^n \text{DG}$  schemes equipped with a closed embedding into  $\mathcal{X}$ . For an arrow  $(\beta_1 \rightarrow \beta_2) \in \mathbf{B}$ , let  $f_{\beta_1, \beta_2}$  denote the corresponding closed embedding  $X_{\beta_1} \rightarrow X_{\beta_2}$ .

Thus, we obtain a map

$$(5.2) \quad \text{colim}_{\beta \in \mathbf{B}} X_\beta \rightarrow \mathcal{X},$$

and we claim that it is an isomorphism.

In other words, we have to show that for  $S' \in \leq^n \text{DGSch}^{\text{aff}}$ , the maps  $f_\beta$  induce an isomorphism:

$$(5.3) \quad \text{colim}_{\beta \in \mathbf{B}} \text{Maps}(S', X_\beta) \simeq \text{Maps}(S', \mathcal{X}).$$

---

<sup>17</sup>Here again, we are using the notational conventions of Sect. 0.5.4 where we use the symbol  $X_\alpha$  to also denote the object of  $\leq^n \text{DGSch}$  that should properly be denoted by  $\leq^{nL} \text{LKE}_{\leq^{n-1} \text{DGSch}^{\text{aff}}} \hookrightarrow \leq^n \text{DGSch}^{\text{aff}}(X_\alpha)$ .

5.1.5. *Step 3: a map in the opposite direction.* Let us construct a map that we shall eventually prove to be the inverse of (5.3):

$$(5.4) \quad \text{Maps}(S', \mathcal{X}) \rightarrow \text{colim}_{\beta \in \mathbf{B}} \text{Maps}(S', X_\beta).$$

For  $S' \in {}^{\leq n} \text{DGSch}^{\text{aff}}$ , set  $S := {}^{\leq n-1} S'$ . Tautologically, we have:

$$(5.5) \quad \text{Maps}(S', \mathcal{X}) \simeq \text{colim}_{\alpha \in \mathbf{A}} \{x : S \rightarrow X_\alpha, x' : S' \rightarrow \mathcal{X}, x'|_S \simeq f_\alpha \circ x\}.$$

We can view  $S'$  as a square-zero extension of  $S$  by the object

$$H^{-n}(\mathcal{O}_{S'}) \in \text{QCoh}(S)^\heartsuit[n] \subset \text{QCoh}(S)^{\geq -n, \leq 0}.$$

By (4.16) and the fact that  $\mathbf{B}_\alpha := {}^{\leq n} \text{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in } \mathcal{X}}$  is cofinal in  ${}^{\leq n} \text{SqZExt}(X_\alpha, f_\alpha)$ , we have:

$$\begin{aligned} \{x : S \rightarrow X_\alpha, x' : S' \rightarrow \mathcal{X}, x'|_S \simeq f_\alpha \circ x\} &\simeq \\ &\simeq \text{colim}_{\beta \in \mathbf{B}_\alpha} \{x : S \rightarrow X_\alpha, x' : S' \rightarrow X_\beta, x'|_S \simeq i_\beta \circ x\}. \end{aligned}$$

Hence, the expression in (5.5) can be rewritten as

$$(5.6) \quad \text{colim}_{\alpha \in \mathbf{A}} \left( \text{colim}_{\beta \in \mathbf{B}_\alpha} \{x : S \rightarrow X_\alpha, x' : S' \rightarrow X_\beta, x'|_S \simeq i_\beta \circ x\} \right) \simeq \\ \simeq \text{colim}_{\beta \in \mathbf{B}} \{x : S \rightarrow X_\alpha, x' : S' \rightarrow X_\beta, x'|_S \simeq i_\beta \circ x\},$$

The expression in the RHS of (5.6) maps to the RHS of (5.4) by forgetting the data of  $x$  and the isomorphism  $x'|_S = i_\beta \circ x$ .

It is immediate from the construction, the composite arrow

$$\text{Maps}(S', \mathcal{X}) \xrightarrow{(5.4)} \text{colim}_{\beta \in \mathbf{B}} \text{Hom}(S', X_\beta) \xrightarrow{(5.3)} \text{Hom}(S', \mathcal{X})$$

is the identity map.

5.1.6. *Step 4: computation of the other composition.* It remains to show that the composition

$$(5.7) \quad \text{colim}_{\beta \in \mathbf{B}} \text{Maps}(S', X_\beta) \xrightarrow{(5.3)} \text{Maps}(S', \mathcal{X}) \xrightarrow{(5.4)} \text{colim}_{\beta \in \mathbf{B}} \text{Maps}(S', X_\beta)$$

is isomorphic to the identity map.

To do this, we introduce yet another category, denoted  $\Gamma$ . The objects of  $\Gamma$  are the data of

- An arrow  $(\beta \rightarrow \beta_1) \in \mathbf{B}$ , which projects by means of  $\phi$  to an arrow  $(\alpha \rightarrow \alpha_1) \in \mathbf{A}$ ,
- A map  $g_{\beta, \alpha_1} : {}^{\leq n-1} X_\beta \rightarrow X_{\alpha_1}$ ,
- A commutative diagram of square-zero extensions compatible with maps to  $\mathcal{X}$

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta, \beta_1}} & X_{\beta_1} \\ j_\beta \uparrow & & \uparrow i_{\beta_1} \\ {}^{\leq n-1} X_\beta & \xrightarrow{g_{\beta, \alpha_1}} & X_{\alpha_1}, \end{array}$$

where  $j_\beta$  is the canonical map, corresponding to the truncation,

- An identification of the composition

$$X_\alpha \xrightarrow{{}^{\leq n-1} i_\beta} {}^{\leq n-1} X_\beta \xrightarrow{g_{\beta, \alpha_1}} X_{\alpha_1} \text{ with } f_{\alpha, \alpha_1},$$

- A homotopy between the resulting two identifications, making the following diagram commutative:

$$\begin{array}{ccc}
 f_{\alpha_1} \circ g_{\beta, \alpha_1} \circ \leq^{n-1} i_{\beta} & \xrightarrow{\sim} & f_{\beta} \circ j_{\beta} \circ \leq^{n-1} i_{\beta} \simeq f_{\beta} \circ i_{\beta} \\
 \sim \downarrow & & \sim \downarrow \\
 f_{\alpha_1} \circ f_{\alpha, \alpha_1} & \longrightarrow & f_{\alpha}.
 \end{array}$$

We can depict this data in a diagram:

$$(5.8) \quad
 \begin{array}{ccccc}
 & & X_{\beta} & \xrightarrow{f_{\beta, \beta_1}} & X_{\beta_1} \\
 & & \uparrow j_{\beta} & & \uparrow i_{\beta_1} \\
 & \leq^{n-1} X_{\beta} & & & \\
 & \uparrow \leq^{n-1} i_{\beta} & \searrow g_{\beta, \alpha_1} & & \searrow f_{\beta} \\
 X_{\alpha} & \xrightarrow{f_{\alpha, \alpha_1}} & X_{\alpha_1} & \xrightarrow{f_{\alpha_1}} & \mathcal{X}.
 \end{array}$$

Morphisms in  $\Gamma$  are defined naturally (so that the corresponding diagrams of schemes commute).

There are tautological maps  $\psi, \psi_1 : \Gamma \rightarrow \mathbf{B}$  that remember the data of  $\beta$  and  $\beta_1$ , respectively.

Consider the colimit

$$\operatorname{colim}_{\gamma \in \Gamma} \operatorname{Maps}(S', X_{\psi(\gamma)}),$$

which admits a tautological map

$$(5.9) \quad r : \operatorname{colim}_{\gamma \in \Gamma} \operatorname{Maps}(S', X_{\psi(\gamma)}) \rightarrow \operatorname{colim}_{\beta \in \mathbf{B}} \operatorname{Maps}(S', X_{\beta}).$$

Note, however, that we have another map

$$(5.10) \quad r_1 : \operatorname{colim}_{\gamma \in \Gamma} \operatorname{Maps}(S', X_{\psi(\gamma)}) \rightarrow \operatorname{colim}_{\beta \in \mathbf{B}} \operatorname{Maps}(S', X_{\beta}),$$

which for  $\gamma \in \Gamma$ , sends  $\operatorname{Maps}(S', X_{\psi(\gamma)})$  to  $\operatorname{Maps}(S', X_{\psi_1(\gamma)})$  by means of  $f_{\psi(\gamma), \psi_1(\gamma)}$ . However, the same edge  $f_{\psi(\gamma), \psi_1(\gamma)}$  provides a homotopy between these two maps of colimits.

It follows from the construction that the composition

$$\operatorname{colim}_{\gamma \in \Gamma} \operatorname{Maps}(S, X_{\psi(\gamma)}) \xrightarrow{r} \operatorname{colim}_{\beta \in \mathbf{B}} \operatorname{Maps}(S', X_{\beta}) \xrightarrow{(5.3)} \operatorname{Maps}(S', \mathcal{X}) \xrightarrow{(5.4)} \operatorname{colim}_{\beta \in \mathbf{B}} \operatorname{Maps}(S', X_{\beta})$$

coincides with the map  $r_1$ .

Therefore, to prove that the composition in (5.7) is isomorphic to the identity map, it suffices to show that the map  $r$  is an isomorphism. To do this, we will repeatedly use the following observation:

**Lemma 5.1.7.** *Let  $F : \mathbf{C}' \rightarrow \mathbf{C}$  be a functor between  $\infty$ -categories.*

- Suppose that  $F$  is a Cartesian fibration. Then  $F$  is cofinal if and only if it has contractible fibers.*
- Suppose that  $F$  is a co-Cartesian fibration, and that  $F$  has contractible fibers. Then it is cofinal.*

It is easy to see that the functor  $\psi$  is a Cartesian fibration. Applying Lemma 5.1.7, we obtain that it is sufficient to show that the fibers of  $\psi$  are contractible.

5.1.8. *Step 5: contractibility of the fibers of  $\psi$ .* For  $\beta \in \mathbf{B}$ , let  $\Gamma_\beta$  denote the fiber of  $\psi$ . Explicitly,  $\Gamma_\beta$  consists of the data of

- An object  $\alpha_1 \in \mathbf{A}$ , and an arrow  $\phi(\beta) =: \alpha \rightarrow \alpha_1$ .
- A map  $g_{\beta, \alpha_1} : \leq^{n-1} X_\beta \rightarrow X_{\alpha_1}$ ,
- An identification of the composition

$$X_\alpha \xrightarrow{\leq^{n-1} i_\beta} \leq^{n-1} X_\beta \xrightarrow{g_{\beta, \alpha_1}} X_{\alpha_1} \text{ with } f_{\alpha, \alpha_1},$$

- A homotopy between the resulting two identifications, making the following diagram commutative:

$$\begin{array}{ccc} f_{\alpha_1} \circ g_{\beta, \alpha_1} \circ \leq^{n-1} i_\beta & \xrightarrow{\sim} & f_\beta \circ j_\beta \circ \leq^{n-1} i_\beta \simeq f_\beta \circ i_\beta \\ \sim \downarrow & & \sim \downarrow \\ f_{\alpha_1} \circ f_{\alpha, \alpha_1} & \longrightarrow & f_\alpha. \end{array}$$

- A lift of  $\alpha \rightarrow \alpha_1$  to an arrow  $\beta \rightarrow \beta_1$ .
- A commutative diagram of square-zero extensions compatible with maps to  $\mathcal{X}$

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta, \beta_1}} & X_{\beta_1} \\ j_\beta \uparrow & & \uparrow i_{\beta_1} \\ \leq^{n-1} X_\beta & \xrightarrow{g_{\beta, \alpha_1}} & X_{\alpha_1}. \end{array}$$

We introduce the category  $\Delta_\beta$  to consist of the first four out of six of the pieces of data in the description of  $\Gamma_\beta$  given above. I.e., an object of  $\Delta_\beta$  corresponds to a diagram

$$(5.11) \quad \begin{array}{ccccc} & & X_\beta & & \\ & & \uparrow j_\beta & & \\ & & \leq^{n-1} X_\beta & & \\ & & \uparrow \leq^{n-1} i_\beta & & \\ X_\alpha & \xrightarrow{f_{\alpha, \alpha_1}} & X_{\alpha_1} & \xrightarrow{f_{\alpha_1}} & \mathcal{X}. \end{array}$$

$\searrow f_\beta \nearrow$

We have a natural forgetful map  $\Gamma_\beta \rightarrow \Delta_\beta$ . It is easy to see that this functor is a co-Cartesian fibration. Hence, by Lemma 5.1.7, it is enough to show that  $\Delta_\beta$  is contractible, and that the fibers of  $\Gamma_\beta$  over  $\Delta_\beta$  are contractible.

5.1.9. *Step 6: contractibility of  $\Delta_\beta$ .* By construction, we have a left fibration  $\Delta_\beta \rightarrow \mathbf{A}_{\alpha/}$ , and hence the homotopy type of  $\Delta_\beta$  is

$$\operatorname{colim}_{\alpha_1 \in \mathbf{A}_{\alpha/}} \left( \operatorname{Maps}(\leq^{n-1} X_\beta, X_{\alpha_1}) \times_{\operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})} \times_{\operatorname{Maps}(X_\alpha, \mathcal{X})} \operatorname{Maps}(X_\alpha, X_{\alpha_1}) \operatorname{pt} \right),$$

where  $\operatorname{pt} \rightarrow \operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})$  is the map  $f_\beta \circ j_\beta$  and  $\operatorname{pt} \rightarrow \operatorname{Maps}(X_\alpha, X_{\alpha_1})$  is  $f_{\alpha, \alpha_1}$ .

Since the category  $A_{\alpha/}$  of objects  $\alpha_1 \in \mathbf{A}$  under  $\alpha$  is filtered, we can commute the colimit and the Cartesian products, and we obtain that the homotopy type of  $\Delta_\beta$  is

$$\left( \operatorname{colim}_{\alpha_1 \in A_{\alpha/}} \operatorname{Maps}(\leq^{n-1} X_\beta, X_{\alpha_1}) \right)_{\operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})} \times_{\operatorname{Maps}(X_\alpha, \mathcal{X})} \left( \operatorname{colim}_{\alpha_1 \in A_{\alpha/}} \operatorname{Maps}(X_\alpha, X_{\alpha_1}) \right) \operatorname{pt}.$$

Since the DG schemes  $X_\alpha$  and  $X_\beta$  are quasi-compact and quasi-separated, the maps

$$\operatorname{colim}_{\alpha_1 \in A_{\alpha/}} \operatorname{Maps}(\leq^{n-1} X_\beta, X_{\alpha_1}) \simeq \operatorname{colim}_{\alpha_1 \in \mathbf{A}} \operatorname{Maps}(\leq^{n-1} X_\beta, X_{\alpha_1}) \rightarrow \operatorname{Maps}(\leq^{n-1} X_\beta, \leq^{n-1} \mathcal{X})$$

and

$$\operatorname{colim}_{\alpha_1 \in A_{\alpha/}} \operatorname{Maps}(X_\alpha, X_{\alpha_1}) \simeq \operatorname{colim}_{\alpha_1 \in \mathbf{A}} \operatorname{Maps}(X_\alpha, X_{\alpha_1}) \rightarrow \operatorname{Maps}(X_\alpha, \leq^{n-1} \mathcal{X}) \simeq \operatorname{Maps}(X_\alpha, \mathcal{X})$$

are isomorphisms.

We obtain that the homotopy type of  $\Delta_\beta$  is

$$\begin{aligned} \operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})_{\operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})} \times_{\operatorname{Maps}(X_\alpha, \mathcal{X})} \operatorname{Maps}(X_\alpha, \mathcal{X}) \operatorname{pt} &\simeq \\ &\simeq \operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})_{\operatorname{Maps}(\leq^{n-1} X_\beta, \mathcal{X})} \times \operatorname{pt} \simeq \operatorname{pt}. \end{aligned}$$

5.1.10. *Step 7: contractibility of the fibers*  $\Gamma_\beta \rightarrow \Delta_\beta$ . For an object  $\delta_\beta \in \Delta_\beta$  as above, the fiber of  $\Gamma_\beta$  over it is the category of

- $\beta_1 \in \mathbf{B}_{\alpha_1}$ ,
- A map of square-zero extensions:

$$\begin{array}{ccc} X_\beta & \xrightarrow{f_{\beta, \beta_1}} & X_{\beta_1} \\ j_\beta \uparrow & & \uparrow i_{\beta_1} \\ \leq^{n-1} X_\beta & \xrightarrow{g_{\beta, \alpha_1}} & X_{\alpha_1}, \end{array}$$

compatible with the maps to  $\mathcal{X}$ .

Let  $j : Z \hookrightarrow Z'$  be any square-zero extension in  $\leq^n \operatorname{DGSch}$ , and let  $x_1 : Z \rightarrow X_{\alpha_1}$ ,  $x' : Z' \rightarrow \mathcal{X}$  be fixed maps equipped with an identification

$$f_{\alpha_1} \circ x_1 \simeq x' \circ j.$$

(In our case, we are going to take  $Z = \leq^{n-1} X_\beta$  and  $Z' = X_{\beta.}$ ) Consider the category of pairs:

- $\beta_1 \in \mathbf{B}_{\alpha_1}$ ,
- A map of square-zero extensions

$$\begin{array}{ccc} Z' & \xrightarrow{x'_1} & X_{\beta_1} \\ j \uparrow & & \uparrow i_{\beta_1} \\ Z & \xrightarrow{x_1} & X_{\alpha_1}, \end{array}$$

compatible with the maps to  $\mathcal{X}$

We claim that this category is contractible. Indeed, if we omit the condition of compatibility with the given map  $x' : Z' \rightarrow \mathcal{X}$ , we obtain the category whose homotopy type is

$$\operatorname{colim}_{\beta_1 \in \mathbf{B}_{\alpha_1}} \{\text{maps of square-zero extensions as above}\},$$

which, by the definition of  $\mathbf{B}_{\alpha_1}$ , is homotopy equivalent to

$$\operatorname{Maps}(Z', \mathcal{X}) \times_{\operatorname{Maps}(Z, \mathcal{X})} \operatorname{pt},$$

where the map  $\operatorname{pt} \rightarrow \operatorname{Maps}(Z, \mathcal{X})$  is given by  $f_{\alpha_1} \circ x_1 = x' \circ j$ .

Reinstating the compatibility condition results in taking the Cartesian product

$$\operatorname{colim}_{\beta_1 \in \mathbf{B}_{\alpha_1}} \left( \{\text{maps of square-zero extensions}\} \times_{\operatorname{Maps}(Z', \mathcal{X}) \times_{\operatorname{Maps}(Z, \mathcal{X})} \operatorname{pt}} \operatorname{pt} \right).$$

Since  $\mathbf{B}_{\alpha_1}$  is filtered, the above colimit can be rewritten as

$$\begin{aligned} \operatorname{colim}_{\beta_1 \in \mathbf{B}_{\alpha_1}} \{\text{maps of square-zero extensions}\} \times_{\operatorname{Maps}(Z', \mathcal{X}) \times_{\operatorname{Maps}(Z, \mathcal{X})} \operatorname{pt}} \operatorname{pt} &\simeq \\ &\simeq \left( \operatorname{Maps}(Z', \mathcal{X}) \times_{\operatorname{Maps}(Z, \mathcal{X})} \operatorname{pt} \right) \times_{\operatorname{Maps}(Z', \mathcal{X}) \times_{\operatorname{Maps}(Z, \mathcal{X})} \operatorname{pt}} \operatorname{pt} \simeq \operatorname{pt}. \end{aligned}$$

**5.2. The  $\aleph_0$  condition.** In this subsection we will give a characterization of the  $\aleph_0$  property in terms of cotangent spaces.

5.2.1. Let  $\mathbf{C}$  be a category. We shall say that an object of  $\operatorname{Pro}(\mathbf{C})$  is  $\aleph_0$  if it can be presented as an inverse limit over a category equivalent to  $\mathbb{N}$  as a poset.

5.2.2. Let  $\mathcal{X}$  be an object of  $\leq^n \operatorname{DGindSch}$ . We shall denote by  $\mathcal{X}_{n+1}$  its canonical extension to an object of  $\leq^{n+1} \operatorname{DGindSch}$ , i.e.,

$$\mathcal{X}_{n+1} := {}^{\leq n+1}L \operatorname{LKE}(\mathcal{X}).$$

**Proposition 5.2.3.** *An object  $\mathcal{X} \in \leq^n \operatorname{DGindSch}$  is  $\aleph_0$  if and only if the following two conditions hold:*

(a) *The classical indscheme  ${}^{cl}\mathcal{X}$  is  $\aleph_0$ .*

(b) *The following equivalent conditions hold:*

(i) *There exists a cofinal family of closed embeddings  $x : Z \rightarrow {}^{cl}\mathcal{X}$ , where  $Z \in \operatorname{Sch}_{\text{qc-qs}}$ , such that the object  ${}^{\geq -n-1}(T_x^* \mathcal{X}_{n+1}) \in \operatorname{Pro}(\operatorname{QCoh}(Z)^{\geq -n-1, \leq 0})$  is  $\aleph_0$ .<sup>18</sup>*

(ii) *Same as (i) but for any map  $x : Z \rightarrow {}^{cl}\mathcal{X}$  (i.e., not necessarily a closed embedding).*

(iii) *Same as (ii), but with  $Z$  required to be affine.*

---

<sup>18</sup>Here we are using the notational convention of Sect. 0.5.4 and denote by  $Z$  the object of  $\leq^{-n-1} \operatorname{DGSch}$  obtained as the left Kan extension of  $Z$ .



5.2.4. *Proof of the equivalence of (i), (ii) and (iii).* The implication (ii)  $\Rightarrow$  (i) is tautological. The implication (i)  $\Rightarrow$  (ii) follows from the fact that the formation of  ${}^{\geq -n-1}(T_x^* \mathcal{X}_{n+1})$  is compatible with pull-backs, i.e., Condition (B). The implication (ii)  $\Rightarrow$  (iii) is again tautological. The implication (iii)  $\Rightarrow$  (ii) follows from the next lemma:

**Lemma 5.2.5.** *The equivalence of Lemma 4.2.5 for  $\text{Pro}(\text{QCoh}(-))^{\geq -n, \leq 0}$  preserves the corresponding  $\aleph_0$  subcategories.*

*Proof.* It is easy to see that it is enough to prove the lemma for  $\text{Pro}(\text{QCoh}(-))$  instead of  $\text{Pro}(\text{QCoh}(-))^{\geq -n, \leq 0}$ .

By induction, the assertion reduces to the following statement: let  $Z = Z_1 \cup Z_2$  be a covering of  $Z$  by two Zariski open subsets. Let  $F \in \text{Pro}(\text{QCoh}(Z))$  be an object such that

$$\text{Pro}(j_1^*)(F) \in \text{Pro}(\text{QCoh}(Z_1)) \text{ and } \text{Pro}(j_2^*)(F) \in \text{Pro}(\text{QCoh}(Z_2))$$

are  $\aleph_0$ . Then  $F$  is  $\aleph_0$ .

It is easy to see that if  $F' \rightarrow F'' \rightarrow F'''$  is an exact triangle in  $\text{Pro}(\text{QCoh}(S))$ , then the condition of being  $\aleph_0$  has the “2 out of 3” property. Considering the exact triangle

$$F \rightarrow \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F) \rightarrow \text{Cone}(F \rightarrow \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F)),$$

we obtain that it is sufficient to show that  $\text{Cone}(F \rightarrow \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F))$  is  $\aleph_0$ .

However,  $\text{Cone}(F \rightarrow \text{Pro}(j_{1*}) \circ \text{Pro}(j_1^*)(F))$  is supported on a Zariski-closed subset contained in  $Z_2$  and isomorphic to

$$\text{Cone}(\text{Pro}(j_2^*)(F) \rightarrow \text{Pro}(j_{12,2*}) \circ \text{Pro}(j_{12,2}^*)(F)),$$

(where  $j_{12,2}$  denotes the open embedding  $Z_1 \cap Z_2 \hookrightarrow Z_2$ ), which is  $\aleph_0$  by the “2 out of 3” property. □

This finishes the proof of the equivalence of properties (i), (ii) and (iii) in Proposition 5.2.3. □

5.2.6. *Proof of “the only if” direction.* Suppose  $\mathcal{X}$  is  $\aleph_0$ . Fix its presentation as in (1.1), where the index set  $\mathbf{A}$  is equivalent to  $\mathbb{N}$ . For  $\alpha \in \mathbf{A}$  (resp., for an arrow  $\alpha_1 \rightarrow \alpha_2$ ) let  $f_\alpha$  (resp.,  $f_{\alpha_1, \alpha_2}$ ) denote the corresponding closed embedding  $f_\alpha : X_\alpha \rightarrow \mathcal{X}$  (resp.,  $X_{\alpha_1} \rightarrow X_{\alpha_2}$ ).

For a quasi-compact and quasi-separated  $Z \in {}^{\leq n}\text{DGSch}$  equipped with a map  $x : Z \rightarrow \mathcal{X}$ , let  $\alpha_0$  be an index such that  $x$  factors through a map  $x_{\alpha_0} : Z \rightarrow X_{\alpha_0}$ . By Sect. 4.5.4, we have:

$${}^{\geq -n-1}(T_x^* \mathcal{X}_{n+1}) \simeq \lim_{\alpha \in \mathbf{A}_{\alpha_0/}} {}^{\geq -n-1}(T_{x_{\alpha_0} \circ f_{\alpha_0, \alpha}}^* X_\alpha),$$

and the category of indices is explicitly equivalent to  $\mathbb{N}$ . □

5.2.7. *Proof of “if” direction.* First, we observe that the “2 out of 3” property of an object of  $\text{QCoh}(Z)^{\geq -n-1, \leq 0}$  of being  $\aleph_0$  implies that if conditions (i), (ii) or (iii) hold for  $Z \in \text{Sch}_{\text{qc-cs}}$  equipped with a map to  ${}^{cl}\mathcal{X}$ , then the same will be true for any  $Z \in {}^{\leq n}\text{DGSch}_{\text{qc-cs}}$  equipped with a map to  ${}^{\leq n}\mathcal{X}$ .

By induction, we may assume that the truncation

$${}^{\leq n-1}\mathcal{X} := \mathcal{X}|_{{}^{\leq n-1}\text{DGSch}^{\text{aff}}}$$

is  $\aleph_0$ .

Fix a presentation of  $\leq^{n-1}\mathcal{X}$  as in (1.1), where the category  $\mathcal{A}$  is equivalent to the poset  $\mathbb{N}$ . Consider the corresponding category  $\mathcal{B}$  (see Step 3 in the proof of Theorem 5.1.1), mapping to  $\mathcal{A}$  by means of  $\phi$ . We shall use the following lemma:

**Lemma 5.2.8.** *Let  $\phi : \mathcal{B} \rightarrow \mathcal{A}$  be a co-Cartesian fibration of categories, where  $\mathcal{A}$  is equivalent to  $\mathbb{N}$ , and every fiber admits a cofinal functor from  $\mathbb{N}$ . Then  $\mathcal{B}$  also admits a cofinal functor from  $\mathbb{N}$ .*

Hence, by the lemma, it suffices to show that for each  $\alpha \in \mathcal{A}$ , the corresponding category  $\mathcal{B}_\alpha$  admits a cofinal functor from  $\mathbb{N}$ . By construction, the category

$$\mathcal{B}_\alpha = \leq^n \text{SqZExt}(X_\alpha, f_\alpha)_{\text{closed in } \mathcal{X}}$$

is cofinal in  $\leq^n \text{SqZExt}(X_\alpha, f_\alpha)$ , and the embedding admits a left adjoint. Therefore, it is enough to show that the latter admits a cofinal map from  $\mathbb{N}$ .

We have

$$\leq^n \text{SqZExt}(X_\alpha, f_\alpha) \simeq ((\text{QCoh}(X_\alpha)^{\geq -n, \leq 0})_{\text{Cone}((df_\alpha)^*[-1])})^{op},$$

where  $(df_\alpha)^*$  is the canonical map in  $\text{Pro}(\text{QCoh}(X_\alpha)^{\geq -n-1})$ :

$$\geq^{-n-1}(T_{f_\alpha}^* \mathcal{X}_{n+1}) \rightarrow \geq^{-n-1}(T^* X_\alpha).$$

The assertion now follows from the assumption that  $\geq^{-n-1}(T_{f_\alpha}^* \mathcal{X}_{n+1})$  is  $\aleph_0$ . □

### 5.3. The “locally almost of finite type” condition.

5.3.1. We shall now give a characterization of  $\leq^n \text{DG}$  indschemes locally of finite type in terms of their cotangent spaces.

As before, let  $\mathcal{X}$  be an object of  $\leq^n \text{DGindSch}$ , and set

$$\mathcal{X}_{n+1} := \leq^{n+1} L \text{LKE}(\mathcal{X}).$$

**Proposition 5.3.2.** *The following conditions are equivalent:*

(a)  $\mathcal{X}$  is locally of finite type.

(b)  ${}^{cl}\mathcal{X}$  is locally of finite type and the following equivalent conditions hold:

(i) *There exists a cofinal family of closed embeddings  $x : Z \rightarrow {}^{cl}\mathcal{X}$ , where  $Z \in \text{Sch}_{\text{ft}}$ , such that the object*

$$\geq^{-n-1}(T_x^* \mathcal{X}_{n+1}) \in \text{Pro}(\text{QCoh}(Z)^{\geq -n-1, \leq 0})$$

*belongs to  $\text{Pro}(\text{Coh}(Z)^{\geq -n-1, \leq 0})$ .*

(ii) *Same as (i) but for any map  $x : Z \rightarrow {}^{cl}\mathcal{X}$  (i.e., not necessarily a closed embedding).*

(iii) *Same as (ii), but with  $Z$  required to be affine.*

5.3.3. *Proof of the equivalence of (i), (ii), and (iii).* This is similar to Sect. 5.2.4, using the following interpretation of

$$\text{Pro}(\text{Coh}(Z)^{\geq -n, \leq 0}) \subset \text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$$

for  $Z \in \text{DGSch}_{\text{aft}}$ :

**Lemma 5.3.4.** *For a Noetherian scheme  $Z$ , an object  $F \in \text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$  belongs to  $\text{Pro}(\text{Coh}(Z)^{\geq -n, \leq 0})$  if and only if, when viewed as a functor*

$$F : \text{QCoh}(Z)^{\geq -n, \leq 0} \rightarrow \text{Vect},$$

*it commutes with filtered colimits.*

5.3.5. *Proof of Proposition 5.3.2.* The implication (a)  $\Rightarrow$  (b) follows using Lemma 5.3.4 from the fact that an object of  ${}^{\leq n+1}\text{PreStk}$ , which is locally of finite type, takes limits in  ${}^{\leq n+1}\text{DGSch}^{\text{aff}}$  to colimits, see [GL:Stacks], Corollary 1.3.8.

Let us show that (b) implies (a). By induction, we can assume that  ${}^{\leq n-1}\mathcal{X} := \mathcal{X}|_{{}^{\leq n}\text{DGSch}^{\text{aff}}}$  is locally of finite type.

We claim now that conditions (i), (ii) and (iii) of Proposition 5.3.2(b) hold for any  $Z \in {}^{\leq n}\text{DGSch}_{\text{ft}}$  mapping to  ${}^{\leq n}\mathcal{X}$  (and not just classical schemes). This follows from the next observation:

**Lemma 5.3.6.** *Let  $Z$  be an object of  $\text{DGSch}_{\text{aft}}^{\text{aff}}$ , and  $F \in \text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$ . Then  $F$  belongs to  $\text{Pro}(\text{Coh}(Z)^{\geq -n, \leq 0})$  if and only if its restriction to  ${}^{\text{cl}}Z$  does.*

(For future use, note that the assertion of Lemma 5.3.6 is valid, with the same proof, for  $\text{Pro}(\text{QCoh}(Z)^{\geq -n, \leq 0})$  and  $\text{Pro}(\text{Coh}(Z)^{\geq -n, \leq 0})$  replaced by the categories  $\text{QCoh}(Z)^{\geq -n, \leq 0}$  and  $\text{Coh}(Z)^{\geq -n, \leq 0}$ , respectively.)

*Proof.* The property of commutation with filtered colimits is enough to check on  $\text{QCoh}(Z)^{\heartsuit}$ , and direct image defines an equivalence  $\text{QCoh}({}^{\text{cl}}Z)^{\heartsuit} \simeq \text{QCoh}(Z)^{\heartsuit}$ .  $\square$

The rest of the proof was carried out in the course of the proof of Proposition 1.6.4 in Sect. 3.5.2. Indeed, the induction step only used square-zero extensions, which are controlled by

$$\text{Cone}((dx)^*)[-1] \in \text{Pro}(\text{Coh}(Z)^{\geq -n, \leq 0})$$

for  $Z \in {}^{\leq n-1}\text{DGSch}_{\text{ft}}$  and a closed embedding  $x : Z \rightarrow {}^{\leq n-1}\mathcal{X}$ .

Namely, we had to use the fact that if  $Z'_\alpha \in {}^{\leq n}\text{DGSch}_{\text{ft}}$  is a filtered family of square-zero extensions of  $Z$  and

$$Z' := \lim_{\alpha} Z'_\alpha,$$

then the map

$$\text{colim}_{\alpha} \{x'_\alpha : Z'_\alpha \rightarrow \mathcal{X}, x'_\alpha|_Z \simeq x\} \rightarrow \{x' : Z' \rightarrow \mathcal{X}, x'|_Z \simeq x\}$$

is an isomorphism in  $\infty\text{-Grpd}$  (or, equivalently, in  $\text{Vect}$ ).

However, if  $Z'_\alpha$  (resp.,  $Z'$ ) are given by  $\mathcal{J}_\alpha \in \text{Coh}(Z)^{\geq -n, \leq 0}$  (resp.,  $\mathcal{J} \in \text{QCoh}(Z)^{\geq -n, \leq 0}$ ), so that

$$\mathcal{J} \simeq \text{colim}_{\alpha} \mathcal{J}_\alpha,$$

we have:

$$\text{colim}_{\alpha} \{x'_\alpha : Z'_\alpha \rightarrow \mathcal{X}, x'_\alpha|_Z \simeq x\} \simeq \text{Maps}(\text{Cone}((dx)^*)[-1], \mathcal{J}_\alpha)$$

and

$$\{x' : Z' \rightarrow \mathcal{X}, x'|_Z \simeq x\} \simeq \text{Maps}(\text{Cone}((dx)^*)[-1], \mathcal{J})$$

$\square$

#### 5.4. Variant: characterization of classical indschemes.

5.4.1. We shall say that an object  $\mathcal{X} \in {}^{cl}\text{PreStk}$  is *classically convergent* if for a classical ring  $A$  the map

$$\text{Maps}(\text{Spec}(A), \mathcal{X}) \rightarrow \lim_n \text{Maps}(\text{Spec}(A/A_{nilp}^n), \mathcal{X})$$

is an isomorphism, where  $A_{nilp}$  denotes the ideal of nilpotent elements.

In other words, we require that

$$\mathcal{X} \in {}^{cl}\text{PreStk} = \text{Func}((\text{Sch}^{\text{aff}})^{op}, \infty\text{-Grpd})$$

map isomorphically to the right Kan extension of its restriction to the full subcategory of  $(\text{Sch}^{\text{aff}})^{op}$  formed by those  $\text{Spec}(A)$ , for which the ideal of nilpotent elements in  $A$  is nilpotent (rather than just nil).

Note that any  $\mathcal{X} \in {}^{cl}\text{PreStk}_{\text{ift}}$  is automatically classically convergent, since the relevant rings  $A$  are Noetherian.

5.4.2. Let  ${}^{red}\text{Sch}^{\text{aff}}$  denote the category of classical reduced affine schemes, and let

$${}^{cl,red}\text{PreStk} := \text{Func}(({}^{red}\text{Sch}^{\text{aff}})^{op}, \infty\text{-Grpd}).$$

We have a natural restriction functor

$${}^{cl}\text{PreStk} \rightarrow {}^{cl,red}\text{PreStk},$$

which we denote by  $\mathcal{Y} \mapsto {}^{red}\mathcal{Y}$ . It has a left adjoint given by left Kan extension followed by  ${}^{cl}L$ .

5.4.3. Consider the full subcategory

$${}^{red}\text{indSch} \subset {}^{cl,red}\text{PreStk}$$

analogous to

$$\text{indSch} \subset {}^{cl}\text{PreStk}.$$

It is easy to see that restriction and its left adjoint

$${}^{cl,red}\text{PreStk} \rightleftarrows {}^{cl}\text{PreStk}$$

send the subcategories  ${}^{red}\text{indSch} \subset {}^{cl,red}\text{PreStk}$  and  $\text{indSch} \subset {}^{cl}\text{PreStk}$  to one another.

Moreover, it is easy to see that  $\mathcal{X}_{red} \in {}^{cl,red}\text{PreStk}$  belongs to  ${}^{red}\text{indSch}$  if and only if  ${}^{cl}L\text{LKE}_{red\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}(\mathcal{X}_{red})$  belongs to  $\text{indSch}$ .

5.4.4. Let  $\mathcal{X}$  be an object of  ${}^{cl}\text{PreStk}$ , and suppose that  ${}^{red}\mathcal{X}$  belongs to  ${}^{red}\text{indSch}$ . The next assertion gives criteria for when  $\mathcal{X} \in \text{indSch}$ .

**Theorem 5.4.5.** *Assume that  $\mathcal{X}$  is classically convergent. Then the following conditions are equivalent:*

- (i)  $\mathcal{X}$  is a classical indscheme.
- (ii) The map

$${}^{cl}L\text{LKE}_{red\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}({}^{red}\mathcal{X}) \rightarrow \mathcal{X}.$$

is a monomorphism<sup>19</sup>, and  $\mathcal{X}$  admits an extension to an object  $\mathcal{X}_1 \in {}^{\leq 1}\text{PreStk}$  that satisfies indscheme-like Conditions (A), (B) and (C).

- (iii)  $\mathcal{X}$  admits an extension to an object  $\mathcal{X}_1 \in {}^{\leq 1}\text{PreStk}$  that satisfies indscheme-like Conditions (A), (B) and (C), and the relative cotangent complex of

$${}^{cl}L\text{LKE}_{red\text{Sch}^{\text{aff}} \hookrightarrow \text{Sch}^{\text{aff}}}({}^{cl,red}\mathcal{X}) \rightarrow \mathcal{X}$$

---

<sup>19</sup>See Sect. 6.2.1 for what this means.

vanishes<sup>20</sup>.

We omit the proof as it repeats the argument of Theorem 5.1.1.

*Remark 5.4.6.* The condition that the map

$${}^{cl}L\mathrm{KE}_{red\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{Sch}^{\mathrm{aff}}}({}^{red}\mathcal{X}) \rightarrow \mathcal{X}$$

be a monomorphism is necessary for  $\mathcal{X}$  to be a classical indscheme. Otherwise, the de Rham prestack  $X_{\mathrm{dR}}$  of a classical scheme  $X$  (see Sect. 6.1.4) provides a counterexample.

## 6. FORMAL COMPLETIONS

### 6.1. The setting.

6.1.1. In this section we will study the following situation. Let  $\mathcal{X}$  be an object of  $\mathrm{PreStk}$ , and let  $\mathcal{Y}$  be an object of

$${}^{cl,red}\mathrm{PreStk} := \mathrm{Funct}(({}^{red}\mathrm{Sch}^{\mathrm{aff}})^{op}, \infty\text{-Grpd}),$$

where  ${}^{red}\mathrm{Sch}^{\mathrm{aff}}$  denotes the category of classical reduced affine schemes. Let  $\mathcal{Y} \rightarrow {}^{cl,red}\mathcal{X}$  be a map, where  ${}^{cl,red}\mathcal{X} := \mathcal{X}|_{{}^{red}\mathrm{Sch}^{\mathrm{aff}}}$ .

**Definition 6.1.2.** *By the formal completion of  $\mathcal{X}$  along  $\mathcal{Y}$ , denoted  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ , we shall mean the object of  $\mathrm{PreStk}$  equal to the Cartesian product*

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{Y}}^{\wedge} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathrm{RKE}_{red\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Y}) & \longrightarrow & \mathrm{RKE}_{red\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{cl,red}\mathcal{X}). \end{array}$$

In plain terms for  $S \in \mathrm{DGSch}^{\mathrm{aff}}$ , we set  $\mathrm{Maps}(S, \mathcal{X}_{\mathcal{Y}}^{\wedge})$  to be the groupoid consisting of pairs  $(x, y)$ , where  $x : S \rightarrow \mathcal{X}$ , and  $y$  is a lift of the map

$$x|_{{}^{cl,red}S} : {}^{cl,red}S \rightarrow {}^{cl,red}\mathcal{X}$$

to a map  $y : {}^{cl,red}S \rightarrow \mathcal{Y}$ .

6.1.3. Several remarks are in order:

- (i) If  $\mathcal{X}$  is convergent, then so is  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ .
- (ii)  $\mathcal{X}$  and  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  have “the same deformation theory”. In particular, if  $\mathcal{X}$  satisfies Conditions (A), (B) or (C), then so does  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ , and for any  $x : S \rightarrow \mathcal{X}_{\mathcal{Y}}^{\wedge}$ , the map

$$T_x^* \mathcal{X} \rightarrow T_x^* \mathcal{X}_{\mathcal{Y}}^{\wedge}$$

is an isomorphism of functors out of  $\mathrm{QCoh}(S)^{\leq 0}$ .

- (iii) The formation of  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  is compatible with filtered colimits in the sense that for a filtered category  $\mathbf{A}$  and functors

$$\mathbf{A} \rightarrow \mathrm{PreStk} : \alpha \mapsto \mathcal{X}_{\alpha} \text{ and } \mathbf{A} \rightarrow {}^{cl,red}\mathrm{PreStk} : \alpha \mapsto \mathcal{Y}_{\alpha},$$

and a natural transformation  $\mathcal{Y}_{\alpha} \rightarrow {}^{cl,red}\mathcal{X}_{\alpha}$ , the resulting map

$$\mathrm{colim}_{\alpha \in \mathbf{A}} (\mathcal{X}_{\alpha})_{\mathcal{Y}_{\alpha}}^{\wedge} \rightarrow \mathcal{X}_{\mathcal{Y}}^{\wedge}$$

---

<sup>20</sup>Since we are dealing with classical prestacks, the relative cotangent is a priori 0-coconnective, i.e., produces objects of  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq 0, \leq 0}) = \mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit})$ .

is an isomorphism, where

$$\mathcal{X} := \operatorname{colim}_{\alpha \in \mathbf{A}} \mathcal{X}_\alpha \text{ and } \mathcal{Y} := \operatorname{colim}_{\alpha \in \mathbf{A}} \mathcal{Y}_\alpha.$$

(iv) For a map  $\mathcal{X}' \rightarrow \mathcal{X}$  in  $\operatorname{PreStk}$ , let  $\mathcal{Y}' := \mathcal{Y} \times_{\operatorname{cl}, \operatorname{red} \mathcal{X}}^{\operatorname{cl}, \operatorname{red} \mathcal{X}'} \mathcal{X}'$ . Then

$$\mathcal{X}'_{\mathcal{Y}'} \simeq \mathcal{X}'_{\mathcal{Y}} \times_{\mathcal{X}} \mathcal{X}'.$$

6.1.4. When defining formal completions, we can take  $\mathcal{Y} \rightarrow \mathcal{X}|_{\operatorname{redSch}^{\operatorname{aff}}}$  to be an arbitrary map.

For example, taking  $\mathcal{X} = \operatorname{pt}$ , we obtain an object of  $\operatorname{PreStk}$  isomorphic to

$$\operatorname{RKE}_{\operatorname{redSch}^{\operatorname{aff}} \hookrightarrow \operatorname{DGSch}^{\operatorname{aff}}}(\mathcal{Y}).$$

The latter object is otherwise known as the “de Rham space of  $\mathcal{Y}$ ” and is denoted  $\mathcal{Y}_{\operatorname{dR}}$ .

## 6.2. Formal completions along monomorphisms.

6.2.1. Let us now assume that the map  $\mathcal{Y} \rightarrow \operatorname{cl}, \operatorname{red} \mathcal{X}$  is a monomorphism. I.e., for  $S \in \operatorname{redSch}^{\operatorname{aff}}$  and a map  $S \rightarrow \operatorname{cl}, \operatorname{red} \mathcal{X}$ , if there exists a lifting  $S \rightarrow \mathcal{Y}$ , the space of such liftings is contractible.

Note that in this case the map  $\mathcal{X}_{\mathcal{Y}}^{\wedge} \rightarrow \mathcal{X}$  is also a monomorphism. In particular, if  $\mathcal{Z}_i \rightarrow \mathcal{X}$ ,  $i = 1, 2$  are maps in  $\operatorname{PreStk}$  that factor through  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ , then the map

$$(6.1) \quad \mathcal{Z}_1 \times_{\mathcal{X}_{\mathcal{Y}}^{\wedge}} \mathcal{Z}_2 \rightarrow \mathcal{Z}_1 \times_{\mathcal{X}} \mathcal{Z}_2$$

is an isomorphism.

6.2.2. The above observation implies that if  $f : \mathcal{Z} \rightarrow \mathcal{X}_{\mathcal{Y}}^{\wedge}$  is a map such that the composition  $\mathcal{Z} \rightarrow \mathcal{X}_{\mathcal{Y}}^{\wedge} \rightarrow \mathcal{X}$  is a closed embedding, then the original map  $f$  is a closed embedding.

*Remark* Note that the converse to the above statement is not true: consider  $\mathcal{X} := \mathbb{A}_{\operatorname{dR}}^1$ , and  $\mathcal{Y} = \operatorname{pt}$ . We have  $\operatorname{cl}, \operatorname{red} \mathcal{X} = \mathbb{A}^1|_{\operatorname{redSch}^{\operatorname{aff}}}$ , and we let  $\mathcal{Y} \rightarrow \operatorname{cl}, \operatorname{red} \mathcal{X}$  be the map corresponding to  $\{0\} \in \mathbb{A}^1$ . Then  $\mathcal{X}_{\mathcal{Y}}^{\wedge} = \operatorname{pt}$ . The tautological map  $\mathcal{X}_{\mathcal{Y}}^{\wedge} \rightarrow \mathcal{X}$  is now

$$\operatorname{pt} \rightarrow \mathbb{A}_{\operatorname{dR}}^1,$$

and it is not a closed embedding: indeed, its base change with respect to  $\mathbb{A}^1 \rightarrow \mathbb{A}_{\operatorname{dR}}^1$  yields  $(\mathbb{A}^1)_{\{0\}}^{\wedge}$  which is not a closed subscheme of  $\mathbb{A}^1$ .

6.2.3. We would like to consider descent for  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ . This is not completely straightforward since the restriction of the fppf topology to  $\operatorname{redSch}^{\operatorname{aff}}$  does not make much sense.

Let

$$(6.2) \quad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \operatorname{cl}, \operatorname{red} \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y}' & \longrightarrow & \operatorname{cl}, \operatorname{red} (L(\mathcal{X})) \end{array}$$

be a Cartesian diagram in  $\operatorname{cl}, \operatorname{red} \operatorname{PreStk}$  in which the horizontal arrows are monomorphisms.

**Lemma 6.2.4.** *Under the above circumstances, the natural map*

$$L(\mathcal{X}_{\mathcal{Y}}^{\wedge}) \rightarrow (L(\mathcal{X}))_{\mathcal{Y}'}^{\wedge},$$

*is an isomorphism.*

*Proof.* Recall that the sheafification functor  $L$  maps monomorphisms into monomorphisms. Therefore both maps

$$L(\mathcal{X}_{\mathcal{Y}}^{\wedge}) \rightarrow L(\mathcal{X}) \text{ and } (L(\mathcal{X}))_{\mathcal{Y}'}^{\wedge} \rightarrow L(\mathcal{X})$$

are monomorphisms. Hence, the map in the lemma is a monomorphism as well. It requires to see that it is essentially surjective.

Thus, let  $x : S \rightarrow L(\mathcal{X})$  be a map that factors through  $(L(\mathcal{X}))_{\mathcal{Y}'}^{\wedge}$ . We wish to show that it factors through  $L(\mathcal{X}_{\mathcal{Y}}^{\wedge})$  as well. Let  $\pi : \tilde{S} \rightarrow S$  be an fpppf cover, such that  $x \circ \pi$  lifts to a map  $\tilde{x} : \tilde{S} \rightarrow \mathcal{X}$ . It suffices to show that  $\tilde{x}|_{cl, red_S}$  factors through  $\mathcal{Y}$ . However,  $\tilde{x}|_{cl, red_S}$  factors through

$$\mathcal{Y}' \times_{cl, red(L(\mathcal{X}))} {}^{cl, red} \mathcal{X},$$

by construction, and the required factorization follows from the fact that the diagram (6.2) is Cartesian.  $\square$

**Corollary 6.2.5.** *If  $\mathcal{X}$  is a stack, then so is  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ .*

6.2.6. From now on, we will assume that the map  $\mathcal{Y} \rightarrow {}^{cl, red} \mathcal{X}$  is a closed embedding. I.e., for  $S \in {}^{red} \text{Sch}^{\text{aff}}$  and a map  $S \rightarrow {}^{cl, red} \mathcal{X}$ , the Cartesian product

$$S \times_{cl, red \mathcal{X}} \mathcal{Y},$$

taken in  ${}^{cl, red} \text{PreStk}$  is representable by a (reduced) closed subscheme of  $S$ .

**6.3. Formal completions of DG indshemes.** The next proposition shows that the procedure of formal completion is a way to generate DG indschemes:

**Proposition 6.3.1.** *Suppose that in the setting of Sect. 6.2.6,  $\mathcal{X}$  is a DG indscheme. Then the formal completion  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  is also a DG indscheme.*

We will give three proofs.

*Proof.* (an overkill)

We shall prove the proposition by applying Theorem 5.1.1. We note that Conditions (A), (B) and (C) hold for  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  because they do for  $\mathcal{X}$ , see Sect. 6.1.3(ii) above. Hence, it remains to show that  ${}^{cl}(\mathcal{X}_{\mathcal{Y}}^{\wedge})$  is a classical indscheme. However, this is obvious, as the latter is the colimit

$$\text{colim}_{Z_{cl} \rightarrow {}^{cl} \mathcal{X}} Z_{cl},$$

taken over the (filtered!) category of closed embeddings that at the reduced level factor through  $\mathcal{Y}$ .  $\square$

Note that using Proposition 5.3.2 and Sect. 6.1.3(ii), the above argument also gives:

**Corollary 6.3.2.** *If  $\mathcal{X}$  is locally almost of finite type, then so is  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$ .*

6.3.3. The second proof of Proposition 6.3.1 comes along with an explicit description of  $\mathcal{X}_y^\wedge$  as a colimit of DG schemes:

For  $\mathcal{X} \in \text{DGindSch}$ , let  $\mathbf{C}$  denote the full subcategory of  $(\text{DGSch}_{\text{qc-qs}})_{\text{closed in } \mathcal{X}}$  that consists of those closed embedding  $Z \rightarrow \mathcal{X}$ , which factor through  $\mathcal{X}_y^\wedge$ . Note that by Sect. 6.2.2, for any  $Z \in \mathbf{C}$ , the resulting map  $Z \rightarrow \mathcal{X}_y^\wedge$  is a closed embedding.

**Proposition 6.3.4.** *The category  $\mathbf{C}$  is filtered, and the map*

$$\text{colim}_{Z \in \mathbf{C}} Z \rightarrow \mathcal{X}_y^\wedge,$$

*is an isomorphism, where the colimit is taken in  $\text{PreStk}$ .*

*Proof.* First, we claim that by Corollary 3.2.5, the category  $\mathbf{C}$  contains push-outs and coproducts, and hence, is filtered. Indeed, if  $Z_i \rightarrow \mathcal{X}$  are maps such that the maps

$$\text{cl,red} Z_i \rightarrow \text{cl,red} \mathcal{X}$$

factor through  $\mathcal{Y}$ , then the same will be true for their push-out or coproduct.

Thus, we have to show that for  $S \in \text{DGSch}^{\text{aff}}$  and a map  $S \rightarrow \mathcal{X}$  that factors through  $\mathcal{Y}$  at the reduced classical level, the category of its factorizations

$$S \rightarrow Z \rightarrow \mathcal{X}$$

for  $Z \in \mathbf{C}$ , is contractible. Indeed, it is easy to see that it is filtered by Proposition 3.2.3 and non-empty by Sect. 3.2.7. □

*Remark.* It is not difficult to see that the category  $\mathbf{C}$  used in the above proof is the same as  $(\text{DGSch}_{\text{qc-qs}})_{\text{closed in } \mathcal{X}_y^\wedge}$ , i.e., the assertion on Sect. 6.2.2 is “if and only if” for  $\mathcal{X}$  a DG indscheme and  $\mathcal{Z} = Z \in \text{DGSch}_{\text{qc-qs}}$ .

Indeed, let  $\overline{Z}$  denote the closure of the image of  $Z$  in  $\mathcal{X}$  (see Sect. 3.2.7). It is enough to show that the map  $Z \times_{\mathcal{X}} \overline{Z} \rightarrow \overline{Z}$  is a closed embedding. However, since  $\overline{Z} \rightarrow \mathcal{X}$  also factors through  $\mathcal{X}_y^\wedge$ , the map

$$Z \times_{\mathcal{X}_y^\wedge} \overline{Z} \rightarrow Z \times_{\mathcal{X}} \overline{Z}$$

is an isomorphism, and the map

$$Z \times_{\mathcal{X}_y^\wedge} \overline{Z} \rightarrow \overline{Z},$$

being a base change of  $Z \rightarrow \mathcal{X}_y^\wedge$ , is a closed embedding, by assumption.

6.3.5. Also note that if  $\mathcal{X}$  is written as in (1.5), then if we set  $Y_\alpha := \mathcal{Y} \cap \text{cl,red} X_\alpha$ , by Sect. 6.1.3 (iii) and (iv), we have:

$$\mathcal{X}_y^\wedge \simeq \text{colim}_\alpha (X_\alpha)_{Y_\alpha}^\wedge,$$

where the colimit is taken in  $\text{PreStk}$ .

This gives a third proof of Proposition 6.3.1. Indeed, it is clear that a filtered colimit of DG indschemes with transition maps being closed embeddings is a DG indscheme. Now, for  $X \in \text{DGSch}_{\text{qc-qs}}$  and  $Y$  a closed subset in  $\text{cl,red} X$ , the formal completion  $X_Y^\wedge$  is a DG indscheme, see Proposition 6.5.5 below.



**6.4. Formal (DG) schemes.** Let us recall the following definition:

**Definition 6.4.1.** A classical indscheme  $\mathcal{X}_{cl}$  is called a formal scheme if  ${}^{red}(\mathcal{X}_{cl})$  is a scheme.  
<sup>21</sup>

In the derived setting, we give the following one:

**Definition 6.4.2.** A DG indscheme  $\mathcal{X}$  is called a formal DG scheme if the underlying classical indscheme  ${}^{cl}\mathcal{X}$  is formal.

We have, tautologically:

**Lemma 6.4.3.** In the situation of Proposition 6.3.1, if  $\mathcal{Y}$  is a scheme, then  $\mathcal{X}_{\mathcal{Y}}^{\wedge}$  is a formal DG scheme.

**6.5. Formal completions of DG schemes.** For the rest of this section we will take  $\mathcal{X}$  to be a DG scheme  $X$ , and  $\mathcal{Y}$  to be a Zariski closed subset  $Y$  of  ${}^{cl,red}X$ . Consider the corresponding formal completion  $X_Y^{\wedge}$ .

In this situation, we shall be always assume  $Y$  is quasi-compact and quasi-separated, in order for  $X_Y^{\wedge}$  to be a DG indscheme according to our definition.

6.5.1. First, we have:

**Proposition 6.5.2.**  $X_Y^{\wedge}$  is a DG indscheme.

We note that Proposition 6.5.2 is not, strictly speaking a consequence of Proposition 6.3.1, since if  $X$  fails to be quasi-compact and quasi-separated, then it is not a DG indscheme in our definition. However, it is easy to see that either of the first two proofs of Proposition 6.3.1 applies in this case as well.

We also note that  ${}^{cl,red}(X_Y^{\wedge}) \simeq Y$ . Hence, we obtain:

**Corollary 6.5.3.**  $X_Y^{\wedge}$  is a formal DG scheme.

6.5.4. In the present situation, we can slightly improve the presentation of  $X_Y^{\wedge}$  given by Proposition 6.3.4:

**Proposition 6.5.5.** As an object of  $\text{PreStk}$ ,  $X_Y^{\wedge}$  is isomorphic to

$$\text{colim}_{Y' \rightarrow X} Y',$$

where the colimit is taken over the category of closed embeddings whose set-theoretic image is  $Y$ .

*Proof.* By Corollary 1.4.5, we know that  $X_Y^{\wedge}$  is isomorphic to

$$\text{colim}_{Z \rightarrow X} Z,$$

where the colimit is taken over the category of closed embeddings  $Z \rightarrow X$  whose image is set-theoretically contained in  $Y$ .

By Lemma 3.5.3, it suffices to show that any such  $Z \rightarrow X$  can be factored as  $Z \rightarrow Y' \rightarrow Z$ , where  $Y' \rightarrow Z$  is a closed embedding whose set-theoretic image is exactly  $Y$ .

Let  $Y'_{can}$  be the reduced closed subscheme of  $X$  corresponding to  $Y$ .

---

<sup>21</sup>Recall that we denote by  $\mathcal{Y} \mapsto {}^{red}\mathcal{Y}$  the functor  ${}^{cl}\text{PreStk} \rightarrow {}^{cl,red}\text{PreStk}$  corresponding to restriction along  ${}^{red}\text{Sch}^{\text{aff}} \rightarrow \text{Sch}^{\text{aff}}$ .

Consider the map  ${}^{cl,red}Z \rightarrow {}^{cl}X$ . The latter canonically factors as  ${}^{cl,red}Z \rightarrow Y'_{can} \rightarrow {}^{cl}X$ . The required  $Y'$  is then given by

$$Z \sqcup_{{}^{cl,red}Z} Y'_{can},$$

where we denote by  ${}^{cl,red}Z$  and  $Y'_{can}$  the DG schemes obtained as Kan extensions of the corresponding classical schemes (see the notational convention of Sect. 0.5.4).  $\square$

6.5.6. Note, however, that in general  $X_Y^\wedge$  will fail to be weakly  $\aleph_0$ , even at the classical level. E.g., we take  $X = \mathbb{A}^\infty := \mathrm{Spec}(k[t_1, t_2, \dots])$  and  $Y = \{0\} \subset \mathbb{A}^\infty$ .

However,  $X_Y^\wedge$  is weakly  $\aleph_0$  under the following additional condition:

**Proposition 6.5.7.** *Assume that  $Y$  can be represented by a subscheme  $Y'$  of  ${}^{cl}X$ , whose ideal is locally finitely generated. Then  $X_Y^\wedge$  is weakly  $\aleph_0$  as a DG indscheme.*

*Remark.* We shall soon see in Proposition 6.6.11 that the assertion of the proposition can be significantly strengthened. We suspect that  $X_Y^\wedge$  is actually  $\aleph_0$  (see Sect. 1.5.4 for the distinction between the two notions); we will prove this when  $X$  is affine in Proposition 6.6.8.

*Proof.* We shall deduce the assertion of the proposition from Proposition 5.2.3.

We note that condition (b) of Proposition 5.2.3 follows from Sect. 6.1.3(ii), as it is satisfied for  $X$ .

It remains to show that the classical indscheme underlying  $X_Y^\wedge$  is  $\aleph_0$ . However, the quasi-compactness hypothesis in  $Y$  and the assumption that the ideal  $\mathcal{I}$  of  $Y'$  is locally finitely generated imply that the subschemes  $Y'_n$  given by  $\mathcal{I}^n$  are cofinal among all subschemes of  $X$  whose underlying set is  $Y$ .  $\square$

**6.6. Formal completions along subschemes of finite codimension.** We continue to study formal completions of the form  $X_Y^\wedge$ , where  $X$  is a DG scheme, and  $Y$  is a Zariski closed subset of  ${}^{cl}X$ , which is quasi-compact and quasi-separated.

We will impose the assumption made in Proposition 6.5.7. Namely, will assume that  $Y$  can be represented by a subscheme  $Y'$  of  ${}^{cl}X$ , whose ideal is locally finitely generated.

We will show that in this case, the behavior of  $X_Y^\wedge$  exhibits some additional favorable features.

6.6.1. First, we shall calculate the most basic example: the formal completion of  $\mathbb{A}^1$  at the point  $\{0\}$ . Namely, we shall prove the following:

**Proposition 6.6.2.** *The natural map*

$$\mathrm{colim}_n \mathrm{Spec}(k[t]/t^n) \rightarrow (\mathbb{A}^1)_{\{0\}}^\wedge,$$

*where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism.*

The statement of the proposition is obvious at the level of the underlying classical prestacks, i.e., when we evaluate both sides on  $\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{DGSch}^{\mathrm{aff}}$ . However, some care is needed in the derived setting.

6.6.3. *Proof of Proposition 6.6.2, Step 1.*

Both sides of the proposition are a priori functors  $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}$ . However, we claim that they both, along with the map between them, naturally upgrade to functors

$$(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-PicGrpd},$$

where  $\infty\text{-PicGrpd}$  is the category of  $\infty$ -Picard groupoids, i.e., connective spectra.

Consider first the functor  $\mathrm{Maps}_{\mathrm{PreStk}}(-, \mathbb{A}^1) : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-Grpd}$  represented by  $\mathbb{A}^1$ . We claim that it naturally upgrades to a functor

$$\mathrm{Maps}_{\mathrm{PreStk}}(-, \mathbb{A}^1) : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-PicGrpd}.$$

This comes from the structure on  $\mathbb{A}^1$  of abelian group object in

$$\mathrm{Sch}^{\mathrm{aff}} \subset \mathrm{DGSch}^{\mathrm{aff}} \subset \mathrm{PreStk}.$$

Consider now the object

$$\mathrm{colim}_n \mathrm{Spec}(k[t]/t^n) \in \mathrm{indSch}.$$

The binomial formula endows the above object with a structure of abelian group object in  $\mathrm{indSch}$ ; hence it has a structure of abelian group object in  $\mathrm{DGindSch}$  as well. We shall denote the resulting functor  $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-PicGrpd}$  by  $\mathrm{Nilp}$ .

In what follows we shall denote the functor  $\mathrm{Maps}_{\mathrm{PreStk}}(-, \mathrm{Spec}(k[t]/t^n))$  by  $\mathrm{nilp}_n$ , and

$$\mathrm{colim}_n \mathrm{nilp}_n = \mathrm{nilp},$$

so that we have:

$$\Omega^\infty(\mathrm{Nilp}) = \mathrm{nilp}.$$

The same construction shows that the map of functors

$$\mathrm{nilp} \rightarrow \mathrm{Maps}_{\mathrm{PreStk}}(-, \mathbb{A}^1)$$

naturally upgrades to a map of functors with values in  $\infty\text{-PicGrpd}$

$$\mathrm{Nilp} \rightarrow \mathrm{Maps}_{\mathrm{PreStk}}(-, \mathbb{A}^1).$$

Consider now the functor  $\mathrm{Maps}_{\mathrm{PreStk}}(-, (\mathbb{A}^1)_{\{0\}}^\wedge)$ . Since  $(\mathbb{A}^1)_{\{0\}}^\wedge \hookrightarrow \mathbb{A}^1$  is a monomorphism and gives rise to subgroups at the level of  $\pi_0$ , we obtain that the functor also naturally upgrades to a functor

$$\mathrm{Maps}_{\mathrm{PreStk}}(-, (\mathbb{A}^1)_{\{0\}}^\wedge) : (\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \infty\text{-PicGrpd},$$

and the natural transformation  $\mathrm{Nilp} \rightarrow \mathrm{Maps}_{\mathrm{DGSch}^{\mathrm{aff}}}(-, \mathbb{A}^1)$  factors canonically as

$$\mathrm{Nilp} \rightarrow \mathrm{Maps}_{\mathrm{PreStk}}(-, (\mathbb{A}^1)_{\{0\}}^\wedge) \rightarrow \mathrm{Maps}_{\mathrm{DGSch}^{\mathrm{aff}}}(-, \mathbb{A}^1).$$

6.6.4. *Step 2.*

To prove the proposition, it suffices to show that for a connective commutative DG algebra  $A$ , the map

$$\pi_i(\mathrm{Nilp}(A)) \rightarrow \pi_i(A)$$

is an isomorphism for  $i \geq 1$ , and that  $\pi_0(\mathrm{Nilp}(A))$  maps isomorphically to the set of nilpotent elements in  $\pi_0(A) = {}^{cl}A$ . Here by  $\pi_i$  for  $i \geq 1$  we mean the homotopy group of the connected component of the corresponding space that contains the point 0. Note that by construction,

$$\pi_i(\mathrm{Nilp}(A)) \simeq \operatorname{colim}_n \pi_i(\mathrm{nilp}_n(A)).$$

We have a long exact sequence

$$\dots \pi_{i+1}(\Omega^\infty(A)) \xrightarrow{\text{power } n} \pi_{i+1}(\Omega^\infty(A)) \rightarrow \pi_i(\mathrm{nilp}_n(A)) \rightarrow \pi_i(\Omega^\infty(A)) \xrightarrow{\text{power } n} \pi_i(\Omega^\infty(A)) \dots$$

In the above formula, we view a connective algebra  $A$  as a connective spectrum, i.e., object of  $\infty\text{-PicGrpd}$ , and  $\Omega^\infty(A)$  denotes the underlying space, i.e., object of  $\infty\text{-Grpd}$ , so that  $\pi_i(A) = \pi_i(\Omega^\infty(A))$ .

The map “power  $n$ ” is induced by the corresponding map  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$ , so that we have a Cartesian square in  $\mathrm{DGSch}$ :

$$\begin{array}{ccc} \mathrm{Spec}(k[t]/t^n) & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \text{power } n \\ \{0\} & \longrightarrow & \mathbb{A}^1. \end{array}$$

However, for  $i \geq 1$  and  $n > 1$ , the map  $\pi_i(\Omega^\infty(A)) \xrightarrow{\text{power } n} \pi_i(\Omega^\infty(A))$  is zero. Indeed, this follows from the fact for any two connective algebras  $A_1$  and  $A_2$ , the canonical map

$$\Omega^\infty(A_1) \times \Omega^\infty(A_2) \rightarrow \Omega^\infty(A_1 \otimes A_2)$$

induces a zero map

$$\pi_i(\Omega^\infty(A_1) \oplus \Omega^\infty(A_2)) \rightarrow \pi_i(\Omega^\infty(A_1 \otimes A_2))$$

for  $i \geq 1$ .

So, for  $i \geq 1$  and every  $n$  we have a short exact sequence

$$0 \rightarrow \pi_{i+1}(\Omega^\infty(A)) \rightarrow \pi_i(\mathrm{nilp}_n(A)) \rightarrow \pi_i(\Omega^\infty(A)) \rightarrow 0.$$

By construction, in the diagram

$$\begin{array}{ccccc} \pi_{i+1}(\Omega^\infty(A)) & \longrightarrow & \pi_i(\mathrm{nilp}_{n'}(A)) & \longrightarrow & \pi_i(\Omega^\infty(A)) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{i+1}(\Omega^\infty(A)) & \longrightarrow & \pi_i(\mathrm{nilp}_{n''}(A)) & \longrightarrow & \pi_i(\Omega^\infty(A)) \end{array}$$

the right vertical map is identity, whereas the left vertical map corresponds to the map  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$  given by raising to the power  $n'' - n'$ , and so vanishes for  $i \geq 0$  and  $n'' > n'$ . This proves the required assertion for  $i \geq 1$ . A similar argument treats the case of  $i = 0$ .  $\square$

6.6.5. We now return to the case of more general  $X$  and  $Y$  as in Sect. 6.6. Assume that the DG scheme  $X$  is eventually coconnective. It is natural to ask whether the same will be true for the DG indscheme  $X_Y^\wedge$ .

Note, however, that asking for a DG indscheme to be eventually coconnective (i.e., eventually coconnective as a stack) is a strong requirement, since it is difficult to satisfy it together with convergence, see [GL:Stacks], Remark 1.2.6.

However, the answer to the above question turns out to be affirmative:

**Proposition 6.6.6.** *If  $X$  is eventually coconnective, then  $X_Y^\wedge$  is eventually coconnective as a DG indscheme.*

6.6.7. In order to prove Proposition 6.6.6, we will give a more explicit description of the formal completion  $X_Y^\wedge$  in the situation of Sect. 6.6 when  $X$  is affine. This description will be handy for the proof of several other assertions in this paper.

Let  $X = \mathrm{Spec}(A)$ , and let  $Y'$  be a closed subscheme of  ${}^{\mathrm{cl}}X$  whose ideal is generated by elements  $\overline{f_1}, \dots, \overline{f_m}$  in

$${}^{\mathrm{cl}}A = H^0(A) = \pi_0(\Omega^\infty(\mathrm{Sp}(A))).$$

Let  $f_1, \dots, f_m$  be points of  $\Omega^\infty(A)$  that project to the  $\overline{f_1}, \dots, \overline{f_m}$ .

For an integer  $n$ , set  $A_n := A[t_{n,1}, \dots, t_{n,m}]$ , where the generators  $t_{n,i}$  are in degree  $-1$ , and  $d(t_{n,i}) = f_i^n$ .

For  $n' \leq n''$  we have a natural map  $A_{n''} \rightarrow A_{n'}$  which is identity on  $A$ , and which sends  $t_{n'',i} \mapsto f_i^{n''-n'} \cdot t_{n',i}$ . We will prove:

**Proposition 6.6.8.** *The natural map*

$$(6.3) \quad \mathrm{colim}_n \mathrm{Spec}(A_n) \rightarrow X_Y^\wedge,$$

where the colimit is taken in  $\mathrm{PreStk}$ , is an isomorphism.

*Proof.* The functions  $f_1, \dots, f_m$  define a map

$$\mathrm{Spec}(A) \rightarrow \mathbb{A}^m,$$

and by definition,  $X_Y^\wedge$  maps isomorphically to the Cartesian product

$$(\mathbb{A}^m)_{\{0\}}^\wedge \times_{\mathbb{A}^m} \mathrm{Spec}(A).$$

Since Cartesian products commute with filtered colimits, from Proposition 6.6.2, we obtain that

$$(\mathbb{A}^m)_{\{0\}}^\wedge \times_{\mathbb{A}^m} \mathrm{Spec}(A)$$

is isomorphic to the colimit over  $n$  of

$$(6.4) \quad (\{0\} \times_{\mathbb{A}^m} \mathbb{A}^m) \times_{\mathbb{A}^m} \mathrm{Spec}(A),$$

where the map  $\mathbb{A}^m \rightarrow \mathbb{A}^m$  is given by raising to the power  $n$  along each coordinate. Now, by definition, the DG scheme in (6.4) is isomorphic to  $\mathrm{Spec}(A_n)$ , as required.  $\square$

6.6.9. Let us show how Proposition 6.6.8 implies Proposition 6.6.6:

*Proof.* (of Proposition 6.6.6)

First, note that the assertion is local in the Zariski topology on  $X$ . Thus, we can assume that  $X = \mathrm{Spec}(A)$  is affine.

Now, the assertion follows from the fact that if  $A$  is  $l$ -coconnective, then each of the algebras  $A_n$  is  $(m+l)$ -coconnective, by construction.  $\square$

6.6.10. Here is another corollary of Proposition 6.6.8:

**Proposition 6.6.11.** *The DG indscheme  $X_Y^\wedge$  can be written as a colimit in  $\mathrm{Stk}$*

$$\operatorname{colim}_{\alpha \in \mathbf{A}} Y'_\alpha,$$

where  $Y'_\alpha \rightarrow X$  are closed embeddings with set-theoretic image equal to  $Y$ , and where the category  $\mathbf{A}$  of indices is equivalent to the poset  $\mathbb{N}$ .

*Remark.* This proposition does not prove that  $X_Y^\wedge$  is  $\aleph_0$ , because the colimit is taken in  $\mathrm{Stk}$  and not  $\mathrm{PreStk}$ .

*Proof.* First, note that Proposition 6.6.8 gives such a presentation if  $X$  is affine (moreover, in this case, the colimit can be taken in  $\mathrm{PreStk}$ ). I.e., in this case,  $X_Y^\wedge$  is  $\aleph_0$  as a DG indscheme.

Let  $S_i$  be a (finite) collection of affine open DG subschemes of  $X$  that covers  $Y$ . For each  $i$ , let  $\mathbf{A}_i$  be the corresponding index set (isomorphic to  $\mathbb{N}$ ) for the formal completion  $(S_i)_{S_i \cap Y}^\wedge$ .

For  $\alpha_i \in \mathbf{A}_i$  let  $Y'_{i, \alpha_i}$  be the corresponding DG scheme equipped with a closed embedding into  $S_i$ . Let  $\overline{Y}'_{i, \alpha_i}$  be the closure of its image in  $X$ , see Sect. 3.1.12.

For  $\alpha := \{i \mapsto (\alpha_i \in \mathbf{A}_i)\}$  set  $Y'_\alpha$  be the coproduct of  $\overline{Y}'_{i, \alpha_i}$  in  $(\mathrm{DGSch}_{\mathrm{qs-qs}})_{\mathrm{closed}}$  in  $X$ .

We claim that the family  $\alpha \mapsto Y'_\alpha$  has the desired property. Indeed, it is sufficient to show that for every  $i$ , the colimit of the family

$$\alpha \mapsto Y'_\alpha \times_X S_i$$

is isomorphic to  $(S_i)_{S_i \cap Y}^\wedge$ . However, this is clear, as the above family is cofinal with the initial family  $i \mapsto Y'_{i, \alpha_i}$ .  $\square$

**6.7. Classical vs. derived formal completions.** We shall now show how Proposition 6.6.8 helps answer another natural question regarding the behavior of  $X_Y^\wedge$ .

6.7.1. Let  $X$  a DG scheme, which is 0-coconnective, i.e., a left Kan extension of a classical scheme.

One can ask whether the DG indscheme  $X_Y^\wedge$  is also 0-coconnective. That is, we consider the classical indscheme  ${}^{cl}(X_Y^\wedge)$ , and let

$$\mathcal{X} := {}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}} ({}^{cl}(X_Y^\wedge)).$$

By adjunction, we obtain a map

$$(6.5) \quad \mathcal{X} \rightarrow X_Y^\wedge,$$

and we wish to know whether it is an isomorphism.

Again, we emphasize that it is a rather strong property for a DG indscheme (or any convergent stack) to be 0-coconnective (rather than weakly 0-coconnective), see [GL:Stacks], Remark 1.2.6.

However, the answer to the above question turns out to be affirmative, under an additional assumption that  $X$  is Noetherian (see [GL:IndCoh], Sect. 0.6.9 for the notion of Noetherianness in the DG setting):

**Proposition 6.7.2.** *For  $X$  is Noetherian, the map (6.5) is an isomorphism.*

*Proof.* The assertion readily reduces to the case when  $X$  is affine;  $X = \text{Spec}(A)$ , where  $A$  is a classical  $k$ -algebra. Let  $f_1, \dots, f_m \in A$  be the generators of the ideal of some subscheme  $Y' \subset X$  whose underlying Zariski-closed subset is  $Y$ . Let  $A_n$  be the algebras as in Proposition 6.6.8.

For each  $n$ , let  $A'_n$  be the classical algebra  $H^0(A_n)$ , so that

$$\mathcal{X} \simeq \text{colim}_n \text{Spec}(A'_n),$$

where the colimit is taken in  $\text{PreStk}$ . We will show that inverse systems  $\{A_n\}$  and  $\{A'_n\}$  are equivalent. This follows from the next assertion:

**Lemma 6.7.3.** *For every  $n$  there exists  $N \geq n$  such that the map  $A_N \rightarrow A_n$  can be factored as*

$$A_N \rightarrow A''_N \rightarrow A_n,$$

where  $A''_N$  is acyclic away from degree 0 and  $H^0(A_N) \rightarrow H^0(A''_N)$  is an isomorphism.

□

6.7.4. *Proof of Lemma 6.7.3.* The assertion of the lemma is equivalent to the following: for every  $n$  there exists  $N \geq n$ , such that the maps

$$H^{-i}(A_N) \rightarrow H^{-i}(A_n)$$

are zero for  $i > 0$ . The latter formulation will follow from the Artin-Rees lemma:

**Lemma 6.7.5.** *Let  $B \rightarrow A$  be a map of (classical) Noetherian rings; let  $I \subset B$  be an ideal, and let  $M$  be a finitely generated  $A$ -module. Then for every  $i > 0$ , the inverse system*

$$n \mapsto \text{Tor}_i^B(A, B/I^n)$$

is equivalent to zero, i.e., for every  $n$  there exists an  $N \geq n$ , such that the map

$$\text{Tor}_i^B(M, B/I^N) \rightarrow \text{Tor}_i^B(M, B/I^n)$$

is zero.

We apply this lemma to  $A$  being our algebra  $A$ ,  $B = k[t_1, \dots, t_m]$ , and  $B \rightarrow A$  being given by  $f_1, \dots, f_m$ . Let  $I \subset B$  be the ideal generated by  $t_1, \dots, t_m$ . Let  $I^n$  be the ideal generated by  $t_1^n, \dots, t_m^n$ . Note that

$$H^{-i}(A_n) \simeq \text{Tor}_i^B(A, B/I^n).$$

Finally, the system of ideals  $n \mapsto I^n$  is cofinal with  $n \mapsto I^n$ .

□

6.7.6. *Exponential map.* Let  $\widehat{\mathbb{G}}_a$  and  $\widehat{\mathbb{G}}_m$  be the formal completions of  $\mathbb{G}_a$  and  $\mathbb{G}_m$  at 0 and 1, respectively. These are both formal group schemes. By Proposition 6.7.2, we have that  $\widehat{\mathbb{G}}_a$  and  $\widehat{\mathbb{G}}_m$  are classical formal schemes. In particular, we have the exponential map

$$\exp : \widehat{\mathbb{G}}_m \rightarrow \widehat{\mathbb{G}}_m,$$

which is an isomorphism of formal groups.

For a DG ring  $A$ , let  $\mathrm{Nilp}(A)$  denote the connective spectrum (i.e.,  $\infty$ -Picard groupoid)

$$\ker(A \rightarrow {}^{cl, red}A).$$

Note that by Proposition 6.6.2, the above definition of  $\mathrm{Nilp}(A)$  agrees with one in Sect. 6.6.3.

Let  $A^\times$  denote the connective spectrum of invertible elements in the  $E_\infty$ -ring spectrum  $A$ , and similarly for  ${}^{cl, red}A$ . Set

$$\mathrm{Unip}(A) := \ker(A^\times \rightarrow {}^{cl, red}A^\times).$$

We obtain that the exponential map defines a functorial isomorphism

$$\exp : \mathrm{Nilp}(A) \rightarrow \mathrm{Unip}(A)$$

of functors  $(\mathrm{DGSch}^{\mathrm{aff}})^{op} \rightarrow \{\text{Connective spectra}\}$ .

## 7. QUASI-COHERENT AND IND-COHERENT SHEAVES ON FORMAL COMPLETIONS

Let  $X, Y$  be as in Sect. 6.5. In this section we will assume that  $Y$  is quasi-compact and quasi-separated.

Let  $U$  be the open DG subscheme of  $X$  equal to the complement of  $Y$ ; let  $j$  denote the corresponding open embedding.

### 7.1. Quasi-coherent sheaves on a formal completion.

7.1.1. We have a pair of mutually adjoint functors

$$j^* : \mathrm{QCoh}(X) \rightleftarrows \mathrm{QCoh}(U) : j_*,$$

which realizes  $\mathrm{QCoh}(U)$  as a localization of  $\mathrm{QCoh}(X)$ . Note, however, that the functor  $j_*$  is a priori non-continuous, since  $j$  is not necessarily quasi-compact.

Let  $\mathrm{QCoh}(X)_Y$  denote the full subcategory of  $\mathrm{QCoh}(X)$  equal to

$$\ker(j^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(U)).$$

Let  $\widehat{i}$  denote the canonical map  $X_Y^\wedge \rightarrow X$ , and consider the corresponding functor

$$\widehat{i}^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X_Y^\wedge).$$

We can ask the following questions:

- (i) Is the composition  $\widehat{i}^* \circ j_* : \mathrm{QCoh}(U) \rightarrow \mathrm{QCoh}(X_Y^\wedge)$  zero?
- (ii) Does the functor  $\widehat{i}^*$  induce an equivalence  $\mathrm{QCoh}(X)_Y \rightarrow \mathrm{QCoh}(X_Y^\wedge)$ ?

We will answer these questions in the affirmative under an additional hypothesis on the pair  $X$  and  $Y$ . We learned the corresponding assertion from J. Lurie.



7.1.2. In this section, we will make the assumption of Sect. 6.6, i.e., that  $Y$  can be represented by a closed subscheme  $Y'$  of  ${}^{\text{cl}}X$ , whose ideal is locally finitely generated.

In this case, the morphism  $j$  is quasi-compact (being an open embedding, it is automatically quasi-separated). In particular, by [GL:QCoh], Proposition 2.1.1, the functor  $j_*$  is continuous and satisfies the base change formula, which immediately implies that the composition

$$\widehat{i}^* \circ j_* : \text{QCoh}(U) \rightarrow \text{QCoh}(X_Y^\wedge)$$

vanishes.

**Proposition 7.1.3.** *Under the above hypothesis, the composite functor*

$$\widehat{i}^* : \text{QCoh}(X)_Y \hookrightarrow \text{QCoh}(X) \xrightarrow{\widehat{i}^*} \text{QCoh}(X_Y^\wedge)$$

*is an equivalence.*

7.1.4. *Proof of Proposition 7.1.3.* The assertion is Zarski-local, so we can assume that  $X = \text{Spec}(A)$  is affine. Let  $f_1, \dots, f_m$  and  $A_n$  be as in the proof of Proposition 6.6.8.

Consider the functor

$$\text{QCoh}(X_Y^\wedge) \rightarrow \text{QCoh}(X)$$

given by direct image  $\widehat{i}_*$  with respect to the morphism  $\widehat{i}$ , i.e., the *right* adjoint of the functor  $\widehat{i}^*$ .<sup>22</sup>

*Warning:* The functor  $\widehat{i}_*$  is not continuous and does not commute with Zariski localization.

We obtain that  $\widehat{i}_*$  and  $\widehat{i}^*$  induce a pair of mutually adjoint functors

$$(7.1) \quad (\text{QCoh}(X))_{\text{QCoh}(U)} \rightleftarrows \text{QCoh}(X_Y^\wedge),$$

where  $(\text{QCoh}(X))_{\text{QCoh}(U)}$  denotes the localization of  $\text{QCoh}(X)$  with respect to  $\text{QCoh}(U)$ , and the latter is mapped in by means of  $j_*$ . To prove the proposition it suffices to show that:

- (a) The functor  $\leftarrow$  in (7.1) is fully faithful, and
- (b) The functor  $\rightarrow$  in (7.1) is conservative.

Assertion (a) is equivalent to the functor  $\widehat{i}_*$  being fully faithful. I.e., we need to show that the adjunction map  $\widehat{i}^* \circ \widehat{i}_* \rightarrow \text{Id}$  is an isomorphism.

Fix an object of  $\text{QCoh}(X_Y^\wedge)$ , thought of as a compatible system of  $A_n$ -modules  $\{\mathcal{F}_n\}$ ; let  $\mathcal{F}$  be its direct image on  $X$ . By definition,

$$\mathcal{F} \simeq \lim_n \mathcal{F}_n,$$

where in the right-hand side, the  $\mathcal{F}_n$ 's are regarded as  $A$ -modules.

We need to show that for every  $n_0$ , the map

$$A_{n_0} \otimes_A \mathcal{F} \rightarrow \mathcal{F}_{n_0}$$

is an isomorphism.

Since  $A_{n_0}$  is compact as an  $A$ -module, we can rewrite the left-hand side as  $\lim_n \left( A_{n_0} \otimes_A \mathcal{F}_n \right)$ , and further as

$$\lim_n \left( (A_{n_0} \otimes_A A_n) \otimes_{A_n} \mathcal{F}_n \right).$$

---

<sup>22</sup>Recall that direct image  $g_*$ , although in general non-continuous, is defined for any morphism  $g : \mathcal{Y}_1 \rightarrow \mathcal{Y}_2$  in  $\text{PreStk}$ , by the adjoint functor theorem.

For  $n \geq n_0$  consider the canonical map  $A_{n_0} \otimes_A A_n \rightarrow A_{n_0}$ , and let  $K_n$  denote its kernel. The required assertion follows from the next claim:

**Lemma 7.1.5.** *For every  $n$  there exists  $N \geq n$ , such that the map  $K_N \rightarrow K_n$  is zero as a map of  $A_N$ -modules.*

*Proof.* Let  $B$  denote the polynomial algebra  $k[t_1, \dots, t_m]$ . We have  $A_n \simeq A \otimes_B B_n$ , as commutative  $k$ -algebras. With respect to this identification, we have

$$K_n \simeq A \otimes_B K_n^B,$$

as  $A_n$ -modules, where  $K_n^B$  denotes the corresponding object for the algebra  $B$ . Hence, it is enough to prove the assertion for  $A$  replaced by  $B$ .

To simplify notation, we will only consider the case when  $m = 1$ , i.e.,  $B = k[t]$ . In this case

$$B_n \otimes_B B_{n_0} \simeq \text{Cone}(t^n : k[t]/t^{n_0} \rightarrow k[t]/t^{n_0}).$$

When  $n \geq n_0$ , the map  $t^n : k[t]/t^{n_0} \rightarrow k[t]/t^{n_0}$  is zero, so  $K_n \simeq k[t]/t^{n_0}[1]$ . For  $n' \geq n$ , the corresponding map  $K_{n'} \rightarrow K_n$  is given by multiplication by  $t^{n'-n}$ . Hence, we can take  $N = 2n$ .  $\square$

To prove (b), we will show that if for  $\mathcal{F} \in \text{QCoh}(X)$  we have  $i^*(\mathcal{F}) = 0$ , then  $\mathcal{F}$  belongs to the essential image of  $j_*$ . Here  $i$  is the embedding  $Y' \rightarrow X$ , where  $Y'$  is some closed DG subscheme of  $X$ , whose underlying subset is  $Y$ . Taking the cone we can assume that  $j^*(\mathcal{F}) = 0$  as well, and we need to show that  $\mathcal{F} = 0$ .

We proceed by induction on the number of generators  $m$  of the ideal. Let  $X_1 \subset X$  be the subscheme defined by the ideal  $(f_1)$ , and let  $i_1 : X_1 \rightarrow X$  be the corresponding closed embedding. By the induction hypothesis, we can assume that  $i_1^*(\mathcal{F}) = 0$ . However, this means that  $f_1 : \mathcal{F} \rightarrow \mathcal{F}$  acts invertibly, i.e.,  $\mathcal{F} \rightarrow (\mathcal{F})_{f_1}$  is an isomorphism, where  $(\mathcal{F})_{f_1}$  denotes the localization of  $\mathcal{F}$  with respect to  $f_1$ . However,  $j^*(\mathcal{F}) = 0$  implies  $(\mathcal{F})_{f_1} = 0$ .

$\square$ (Proposition 7.1.3)

7.1.6. Let us denote by  $\mathbf{e}^{\text{QCoh}}$  the tautological embedding

$$\text{QCoh}(X)_Y \rightarrow \text{QCoh}(X).$$

We note that it admits a (continuous) right adjoint, denoted  $\mathbf{r}^{\text{QCoh}}$ , given by

$$\mathcal{F} \mapsto \text{Cone}(\mathcal{F} \rightarrow j_* \circ j^*(\mathcal{F}))[-1].$$

The adjoint pair  $(\mathbf{e}^{\text{QCoh}}, \mathbf{r}^{\text{QCoh}})$  realizes  $\text{QCoh}(X)_Y$  as a *co-localization* of  $\text{QCoh}(X)$ .

By construction, we have a commutative diagram

$$(7.2) \quad \begin{array}{ccc} \text{QCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^*} & \text{QCoh}(X) \\ \widehat{i}^* \uparrow & & \uparrow \text{Id} \\ \text{QCoh}(X)_Y & \xleftarrow{\mathbf{r}^{\text{QCoh}}} & \text{QCoh}(X), \end{array}$$

where the left vertical arrow is the functor from Proposition 7.1.3.

Hence, we obtain that the functor  $\widehat{i}^* : \text{QCoh}(X) \rightarrow \text{QCoh}(X_Y^\wedge)$ , in addition to having a non-continuous right adjoint  $\widehat{i}_*$ , admits a left adjoint, which we denote by  $\widehat{i}_?$ .

Thus, we can think of  $\text{QCoh}(X_Y^\wedge)$  as both a localization and a co-localization of  $\text{QCoh}(X)$  with respect to the essential image of  $\text{QCoh}(U)$ .

Note that under such circumstances, we have a canonical natural transformation

$$(7.3) \quad \widehat{i}_? \rightarrow \widehat{i}_*.$$

7.1.7. Consider now the *non-continuous* functor

$$\mathrm{QCoh}(X) \xrightarrow{\widehat{i}^*} \mathrm{QCoh}(X_Y^\wedge) \xrightarrow{\widehat{i}_*} \mathrm{QCoh}(X),$$

i.e., the localization functor on  $\mathrm{QCoh}(X)$  with respect to the essential image of  $\mathrm{QCoh}(U)$ .

This functor is called *the functor of formal completion* of a quasi-coherent sheaf along  $Y$ . Its essential image (i.e., the essential image of  $\widehat{i}_*$ ) is referred to as objects of  $\mathrm{QCoh}(X)$  that are *adically-complete* with respect to  $Y$ .

**7.2. Compact generation and duality.** Assume now that the scheme  $X$  is quasi-compact and quasi-separated. It is well-known that if  $Y$  is locally given by a finitely generated ideal, then the category  $\mathrm{QCoh}(X)_Y$  is compactly generated by  $\mathrm{QCoh}(X)_Y \cap \mathrm{QCoh}(X)^{\mathrm{perf}}$ .

Combining this with Proposition 7.1.3 and (7.2) we obtain:

**Corollary 7.2.1.** *The category  $\mathrm{QCoh}(X_Y^\wedge)$  is compactly generated. The compact objects are obtained as images under  $\widehat{i}^*$  of compact objects of  $\mathrm{QCoh}(X)$  that are set-theoretically supported on  $Y$ .*

Recall now the notion of quasi-perfectness, see Sect. 2.1.3. We obtain:

**Corollary 7.2.2.** *For  $X$  and  $Y$  as above, the DG indscheme  $\mathrm{QCoh}(X_Y^\wedge)$  is quasi-perfect.*

Let us recall that being quasi-perfect means by definition that the category  $\mathrm{QCoh}(X_Y^\wedge)$  is compactly generated, and that its compact objects belong to  $\mathrm{QCoh}(X_Y^\wedge)^{\mathrm{perf}}$ .

As was shown in Sect. 2.1.3, the above property implies that there exists a canonical equivalence:

$$(7.4) \quad \mathrm{QCoh}(X_Y^\wedge)^\vee \simeq \mathrm{QCoh}(X_Y^\wedge),$$

characterized by either of the following two properties:

- The canonical anti self-equivalence  $\mathbb{D}_{\mathrm{QCoh}(X_Y^\wedge)} : (\mathrm{QCoh}(X_Y^\wedge)^c)^{op} \rightarrow (\mathrm{QCoh}(X_Y^\wedge)^c)$  is given by the restriction of the functor  $\mathcal{F} \mapsto \mathcal{F}^\vee : ((\mathrm{QCoh}(X_Y^\wedge)^{\mathrm{perf}})^{op} \rightarrow \mathrm{QCoh}(X_Y^\wedge)^{\mathrm{perf}})$ .
- The pairing

$$(7.5) \quad \mathrm{QCoh}(X_Y^\wedge) \otimes \mathrm{QCoh}(X_Y^\wedge) \rightarrow \mathrm{Vect}$$

is given by ind-extension of the pairing

$$\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{QCoh}(X_Y^\wedge)^c \mapsto \Gamma(X_Y^\wedge, \mathcal{F}_1 \otimes_{\mathcal{O}_{X_Y^\wedge}} \mathcal{F}_2) \in \mathrm{Vect}.$$

7.2.3. Note that although the natural transformation (7.3) is not an isomorphism, we have the following:

**Lemma 7.2.4.** *The natural transformation (7.3) it induces an isomorphism when restricted to compact objects of  $\mathrm{QCoh}(X_Y^\wedge)$ .*

*Proof.* This follows from the fact that compact objects of  $\mathrm{QCoh}(X)$  with set-theoretic support on  $Y$  are both left and right orthogonal to the essential image of  $\mathrm{QCoh}(U)$ .  $\square$

7.2.5. Recall that the category  $\mathrm{QCoh}(X)$  is also self-dual. From the description of the functor  $\mathbb{D}_{\mathrm{QCoh}(X_Y^\wedge)}$  we obtain that there exists a canonical isomorphism

$$\mathbb{D}_{\mathrm{QCoh}(X)} \circ \widehat{i}_? \simeq \widehat{i}_? \circ \mathbb{D}_{\mathrm{QCoh}(X_Y^\wedge)} : (\mathrm{QCoh}(X_Y^\wedge)^c)^{op} \rightarrow \mathrm{QCoh}(X)^c.$$

By [GL:DG], Lemma 2.3.3, this implies:

**Corollary 7.2.6.** *Under the identifications*

$$\mathrm{QCoh}(X)^\vee \simeq \mathrm{QCoh}(X) \text{ and } \mathrm{QCoh}(X_Y^\wedge)^\vee \simeq \mathrm{QCoh}(X_Y^\wedge),$$

*the dual of the functor  $\widehat{i}^*$  identifies with  $\widehat{i}_?$ .*

7.2.7. Note under the identifications

$$\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X)^\vee \text{ and } \mathrm{QCoh}(U) \simeq \mathrm{QCoh}(U)^\vee$$

we have  $(j_*)^\vee \simeq j^*$ . This implies that the category  $\mathrm{QCoh}(X)_Y$  is also naturally self-dual, such that the dual of the natural embedding

$$\mathbf{e}^{\mathrm{QCoh}} : \mathrm{QCoh}(X)_Y \rightarrow \mathrm{QCoh}(X)$$

is the functor  $\mathbf{r}^{\mathrm{QCoh}}$ .

By [GL:DG], Lemma 2.3.3, this implies:

$$\mathbb{D}_{\mathrm{QCoh}(X)} \circ \mathbf{e}^{\mathrm{QCoh}} \simeq \mathbf{r}^{\mathrm{QCoh}} \circ \mathbb{D}_{\mathrm{QCoh}(X)_Y} : (\mathrm{QCoh}(X)_Y^c)^{op} \rightarrow \mathrm{QCoh}(X)^c.$$

It follows that:

**Corollary 7.2.8.** *The above self-duality of  $\mathrm{QCoh}(X)_Y$  is compactible with the self-duality of  $\mathrm{QCoh}(X_Y^\wedge)$  via the equivalence of Proposition 7.1.3.*

7.3. **t-structures on  $\mathrm{QCoh}(X_Y^\wedge)$ .** In this subsection we will show that the category

$$\mathrm{QCoh}(X_Y^\wedge) \simeq \mathrm{QCoh}(X)_Y$$

possesses two natural t-structures: for one of them the functor  $\mathbf{e}^{\mathrm{QCoh}}$  (i.e., the *left* adjoint of  $\mathbf{r}^{\mathrm{QCoh}} \simeq \widehat{i}^*$ ) is t-exact, and in another the functor  $\widehat{i}_*$  (i.e., the *right* adjoint of  $\widehat{i}^* \simeq \mathbf{r}^{\mathrm{QCoh}}$ ) is t-exact.

7.3.1. Let us recall the following general paradigm: let  $\mathbf{C}$  be a DG category equipped with a t-structure. Let  $F : \mathbf{C}_1 \hookrightarrow \mathbf{C}$  be a fully faithful functor. Assume that  $F$  admits a left (resp., right) adjoint, denoted  $F^L$  (resp.,  $F^R$ ). We have:

**Lemma 7.3.2.**

- (a) *If the composition  $F \circ F^L$  (resp.,  $F \circ F^R$ ) is right (resp., left) t-exact, then  $\mathbf{C}_1$  has a unique t-structure such that  $F$  is t-exact. With respect to this t-structure, the functor  $F^L$  (resp.,  $F^R$ ) is right (resp., left) t-exact.*
- (b) *If the composition  $F \circ F^L$  (resp.,  $F \circ F^R$ ) is left (resp., right) t-exact, then  $\mathbf{C}_1$  has a unique t-structure such that  $F^L$  (resp.,  $F^R$ ) is t-exact. With respect to this t-structure, the functor  $F$  is left (resp., right) t-exact.*

7.3.3. We will apply point (a) of the lemma (with right adjoints) to  $\mathbf{C} = \mathrm{QCoh}(X)$  and  $\mathbf{C}_1 = \mathrm{QCoh}(X)_Y$ .

Let us first take  $F := \mathbf{e}^{\mathrm{QCoh}}$  and  $F^R := \mathbf{r}^{\mathrm{QCoh}}$ . We obtain that  $\mathrm{QCoh}(X)_Y$  obtains a t-structure, compatible with its embedding into  $\mathrm{QCoh}(X)$ . This t-structure is compatible with filtered colimits (i.e., truncation functors commute with filtered colimits).

We shall refer to this t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$  as the “inductive t-structure”.

7.3.4. We shall now introduce another t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$ .

Recall ([GL:QCoh], Sect. 1.2.3) that for any  $\mathcal{Z} \in \mathrm{PreStk}$ , the category  $\mathrm{QCoh}(\mathcal{Z})$  has a canonical t-structure defined by the following requirement: an object  $\mathcal{F}$  belongs to  $\mathrm{QCoh}(\mathcal{Z})^{\leq 0}$  if and only if for every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  and  $\phi : S \rightarrow \mathcal{Z}$ , we have  $\phi^*(\mathcal{F}) \in \mathrm{QCoh}(S)^{\leq 0}$ . Let us call it “the canonical t-structure on  $\mathrm{QCoh}(\mathcal{Z})$ ”.

**Proposition 7.3.5.** *The functor*

$$\widehat{i}_* : \mathrm{QCoh}(X_Y^\wedge) \rightarrow \mathrm{QCoh}(X)$$

*is t-exact for the canonical t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$ .*

A few remarks are in order:

(i) Since the functor  $\widehat{i}^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X_Y^\wedge)$  is right t-exact, we obtain that the proposition implies that the localization functor

$$\widehat{i}_* \circ \widehat{i}^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X)$$

is also right t-exact. Thus, the canonical t-structure on  $\mathrm{QCoh}(\mathcal{Z})$  falls into the paradigm of Lemma 7.3.2(a) with left adjoints.

(ii) The canonical t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$  really is different from the one of Sect. 7.3.3: for the former the functor  $\widehat{i}^*$  is right t-exact, and for the latter it is left t-exact.

(iii) Let  $\mathcal{F}$  be an object of  $\mathrm{QCoh}(X)^\heartsuit$  which is *scheme-theoretically* supported on some subscheme  $Y' \subset X$  whose underlying set is  $Y$ . Then it is easy to see that  $\mathcal{F}$ , regarded as an object of  $\mathrm{QCoh}(X)_Y$ , lies in the heart of both t-structures.

(iv) The canonical t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$  is typically not compatible with colimits, as can be seen in the example of  $X = \mathbb{A}^1$  and  $Y = \mathrm{pt}$ .

*Proof.* (of Proposition 7.3.5)

The functor  $\widehat{i}_*$  is left t-exact, being the right adjoint of a right t-exact functor, namely,  $\widehat{i}^*$ . Hence, we need to show that  $\widehat{i}_*$  is right t-exact.

Let  $\mathcal{Y}$  be an object of  $\mathrm{Stk}$ , and let  $f : \mathcal{Y} \rightarrow X$  be a morphism, where  $X \in \mathrm{DGSch}$ . Assume that  $\mathcal{Y}$  is written as a colimit in  $\mathrm{Stk}$

$$\mathrm{colim}_{g_\alpha : Y'_\alpha \rightarrow \mathcal{Y}} Y'_\alpha,$$

where  $Y'_\alpha \in \mathrm{DGSch}$ . In this case, the functor

$$\mathrm{QCoh}(\mathcal{Y}) \rightarrow \lim_{\alpha} \mathrm{QCoh}(Y'_\alpha)$$

is an equivalence (this follows from [GL:QCoh], Corollary 1.3.7 and the fact that the functor  $\mathrm{QCoh}(-)$  takes colimits in  $\mathrm{PreStk}$  to limits in  $\mathrm{DGCat}$ ).

This implies that the (non-continuous) functor  $f_* : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{QCoh}(X)$  can be calculated as follows: for  $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ , given as a compatible family  $\mathcal{F}_\alpha := g_\alpha^*(\mathcal{F}) \in \mathrm{QCoh}(Y'_\alpha)$ ,

$$f_*(\mathcal{F}) \simeq \lim_{\alpha} (f \circ g_\alpha)_*(\mathcal{F}_\alpha).$$

We apply this to  $\mathcal{Y} := X_Y^\wedge$  written as a colimit as in Proposition 6.6.11. Thus, in order to show that  $\widehat{i}_*$  is right t-exact, we need to check that if  $\mathcal{F}_\alpha \in \mathrm{QCoh}(Y'_\alpha)^{\leq 0}$  for all  $\alpha \in \mathbf{A}$ , then so is

$$\lim_{\alpha \in \mathbf{A}} (i_\alpha)_*(\mathcal{F}_\alpha),$$

where  $i_\alpha$  denotes the map  $Y'_\alpha \rightarrow X$ .

Since the index category is  $A$  is  $\mathbb{N}$ , we need to show that the functor  $\limproj^1$  applied to the family  $\alpha \mapsto H^0((i_\alpha)_*(\mathcal{F}_\alpha))$  vanishes. However, this is the case, since the maps in this family are surjective.  $\square$

**7.4. Ind-coherent sheaves on formal completions.** Let now  $X$  be a DG scheme almost of finite type; in particular, it is quasi-compact (and automatically quasi-separated).

7.4.1. Recall ([GL:IndCoh], Sect. 4.1) that we have a pair of mutually adjoint functors

$$j^{\text{IndCoh},*} : \text{IndCoh}(X) \rightleftarrows \text{IndCoh}(U) : j_*^{\text{IndCoh}}$$

that realize  $\text{IndCoh}(U)$  as a localization of  $\text{IndCoh}(X)$ . Let  $\text{IndCoh}(X)_Y \subset \text{IndCoh}(X)$  be the full subcategory equal to

$$\ker(j^{\text{IndCoh},*}) : \text{IndCoh}(X) \rightarrow \text{IndCoh}(U).$$

We let  $\mathbf{e}^{\text{IndCoh}}$  denote the tautological embedding

$$\text{IndCoh}(X)_Y \hookrightarrow \text{IndCoh}(X).$$

This functor admits a right adjoint, denoted  $\mathbf{r}^{\text{IndCoh}}$  given by

$$\mathcal{F} \mapsto \text{Cone}(\mathcal{F} \rightarrow j_*^{\text{IndCoh}} \circ j^{\text{IndCoh},*}(\mathcal{F}))[-1].$$

7.4.2. As was shown in Corollary 6.3.2, for a Zariski-closed subset  $Y$ , the DG indscheme  $X_Y^\wedge$  is locally almost of finite type, so  $\text{IndCoh}(X_Y^\wedge)$  is well-defined.

Consider the functor <sup>23</sup>

$$\widehat{i}^! : \text{IndCoh}(X) \rightarrow \text{IndCoh}(X_Y^\wedge),$$

i.e., the  $!$ -pullback functor with respect to the morphism  $\widehat{i} : X_Y^\wedge \rightarrow X$ . It is easy to see that this functor annihilates the essential image of  $\text{IndCoh}(U)$  under  $j_*^{\text{IndCoh}}$ .

7.4.3. We now claim that the functor  $\widehat{i}^!$  admits a left adjoint, that shall be denoted by  $\widehat{i}_*^{\text{IndCoh}}$ .

Indeed, by Sect. 2.4.1, we have:

$$(7.6) \quad \text{IndCoh}(X_Y^\wedge) \simeq \text{colim}_\alpha \text{IndCoh}(Y_\alpha),$$

where  $Y_\alpha$  run over a family of closed DG subschemes of  $X$  with the underlying set contained in  $Y$ . If we denote by  $i_\alpha$  the closed embedding  $Y_\alpha \hookrightarrow X$ , the functor  $\widehat{i}_*^{\text{IndCoh}}$ , left adjoint to  $\widehat{i}^!$ , is given by the compatible family of functors

$$(i_\alpha)_*^{\text{IndCoh}} : \text{IndCoh}(Y_\alpha) \rightarrow \text{IndCoh}(X).$$

---

<sup>23</sup>The usage of notation  $\widehat{i}^!$  here is different from [GL:IndCoh], Corollary 4.1.3. Nevertheless, this notation is consistent as will follow from Proposition 7.4.5.

7.4.4. By construction the essential image of the functor  $\widehat{i}_*^{\text{IndCoh}}$  belongs to

$$\text{IndCoh}(X)_Y \subset \text{IndCoh}(X).$$

(Or, equivalently, the right adjoint  $\widehat{i}^!$  of  $\widehat{i}_*^{\text{IndCoh}}$  factors through the co-localization functor  $\mathbf{r}^{\text{IndCoh}}.$ )

Let

$$(7.7) \quad \widehat{i}_*^{\text{IndCoh}} : \text{IndCoh}(X_Y^\wedge) \rightleftarrows \text{IndCoh}(X)_Y : \widehat{i}^!$$

denote the resulting pair of adjoint functors.

We will show:

**Proposition 7.4.5.** *The adjoint functors of (7.7) are equivalences.*

*Proof.* As in the proof of Proposition 7.1.3, we need to show two things:

- (a) The functor  $\widehat{i}_*^{\text{IndCoh}} : \text{IndCoh}(X_Y^\wedge) \rightarrow \text{IndCoh}(X)$  is fully faithful.
- (b) The category  $\text{IndCoh}(X)$  is generated by the essential images of  $\text{IndCoh}(X_Y^\wedge)$  under  $\widehat{i}_*^{\text{IndCoh}}$  and  $\text{IndCoh}(U)$  under  $j_*^{\text{IndCoh}}$ .

The assertion is Zariski-local, so we can assume  $X = \text{Spec}(A)$ . Let  $A_n$  be as in the proof of Proposition 7.1.3. Set  $Y_n := \text{Spec}(A_n)$ .

For  $n' \leq n''$ , let  $i_{n',n''}$  denote the closed embedding  $Y_{n'} \rightarrow Y_{n''}$ , and  $i_n$  the closed embedding  $Y_n \rightarrow X$ . To prove (a), we need to show that for an index  $n_0$  and  $\mathcal{F} \in \text{IndCoh}(Y_{n_0})$ , the map

$$\text{colim}_{n \geq n_0} i_{n_0,n}^! \circ (i_{n_0,n})_*^{\text{IndCoh}}(\mathcal{F}) \rightarrow i_{n_0}^! \circ (i_{n_0})_*^{\text{IndCoh}}(\mathcal{F})$$

is an isomorphism. However, this repeats the computation done in the proof of (a) in Proposition 7.1.3.

To prove (b), it suffices to show that for  $\mathcal{F} \in \text{Coh}(X)^\heartsuit$ , the object

$$\text{Cone}(\mathcal{F} \rightarrow j_*^{\text{IndCoh}} \circ j^{\text{IndCoh},*}(\mathcal{F})) \in \text{IndCoh}(X)^{\geq 0},$$

is a colimit of a diagram of objects, each of which belongs to the essential image of  $\text{Coh}(Y_n)^{\geq 0}$  under  $(i_n)_*^{\text{IndCoh}} : \text{IndCoh}(Y_n) \rightarrow \text{IndCoh}(X)$ . By [GL:IndCoh], Proposition 1.2.2, we can replace  $\text{IndCoh}(-)^+$  by  $\text{QCoh}(-)^+$ , and the assertion becomes manifest.  $\square$

7.4.6. We can depict the assertion of Proposition 7.4.5 as the following commutative diagram, analogous to (7.2):

$$(7.8) \quad \begin{array}{ccc} \text{IndCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^!} & \text{IndCoh}(X) \\ \widehat{i}^! \uparrow & & \uparrow \text{Id} \\ \text{IndCoh}(X)_Y & \xleftarrow{\mathbf{r}^{\text{IndCoh}}} & \text{IndCoh}(X), \end{array}$$

7.4.7. *Compatibility with the  $t$ -structure.* Recall from Sect. 2.4.9, that the category  $\mathrm{IndCoh}(X_Y^\wedge)$  has a natural  $t$ -structure. Note that the category  $\mathrm{IndCoh}(X)_Y$  also has a natural  $t$ -structure for which the functor  $\mathbf{e}^{\mathrm{IndCoh}}$  is  $t$ -exact. Indeed, this follows by Lemma 7.3.2(a) from the fact that the functor

$$\mathbf{e}^{\mathrm{IndCoh}} \circ \mathbf{r}^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X)$$

is left  $t$ -exact.

We claim:

**Lemma 7.4.8.** *The equivalences (7.7) are  $t$ -exact.*

*Proof.* By Lemma 7.3.2(a) we only have to show that the functor

$$\widehat{i}_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X_Y^\wedge) \rightarrow \mathrm{IndCoh}(X)$$

is  $t$ -exact. However, this follows from the description of this functor given in Sect. 7.4.3.  $\square$

## 7.5. Comparison of $\mathrm{QCoh}$ and $\mathrm{IndCoh}$ on a formal completion.

7.5.1. Recall ([GL:IndCoh], Sect. 9.3.2) that for any  $\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{laft}}$  we have a canonical functor

$$\Psi_{\mathcal{Y}}^\vee : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{IndCoh}(\mathcal{Y}),$$

given by tensoring with the dualizing object  $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ .

By construction, the following diagram of functors commutes:

$$(7.9) \quad \begin{array}{ccc} \mathrm{QCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^*} & \mathrm{QCoh}(X) \\ \Psi_{X_Y^\wedge}^\vee \downarrow & & \downarrow \Psi_X^\vee \\ \mathrm{IndCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^\dagger} & \mathrm{IndCoh}(X) \end{array}$$

7.5.2. Also recall that if  $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$  is a DG scheme  $Z \in \mathrm{DGSch}_{\mathrm{laft}}$ , then the category  $\mathrm{IndCoh}(Z)$  is self-dual, and the functor  $\Psi_Z^\vee$  is the dual of the canonically defined functor

$$\Psi_Z : \mathrm{IndCoh}(Z) \rightarrow \mathrm{QCoh}(Z),$$

see [GL:IndCoh], Sect. 1.1.5. However, the functor  $\Psi_{\mathcal{Z}}$  is *not* intrinsically defined when  $\mathcal{Z} \in \mathrm{PreStk}_{\mathrm{laft}}$ .

Nevertheless, for  $\mathcal{X} \in \mathrm{DGindSch}$ , we still have a canonical self-duality

$$\mathrm{IndCoh}(\mathcal{X})^\vee \simeq \mathrm{IndCoh}(\mathcal{X})$$

(see Corollary 2.4.4), and if  $\mathcal{X}$  is quasi-perfect (see Sect. 2.1.3), then we also have a self-duality

$$\mathrm{QCoh}(\mathcal{X})^\vee \simeq \mathrm{QCoh}(\mathcal{X}).$$

So, in this case, we can consider the functor  $\Psi_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{QCoh}(\mathcal{X})$ , dual to  $\Psi_{\mathcal{X}}^\vee$ .

Note the following feature of the resulting functor

$$(7.10) \quad \mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\mathrm{Id} \otimes \Psi_{\mathcal{X}}} \mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{Vect},$$

where the last arrow is the pairing corresponding to the self-duality of  $\mathrm{QCoh}(\mathcal{X})$ .

By construction and Corollary 2.4.8, the functor in (7.10) is isomorphic to the composition

$$\mathrm{QCoh}(\mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{IndCoh}(\mathcal{X}) \xrightarrow{\Gamma^{\mathrm{IndCoh}(\mathcal{X}, -)}} \mathrm{Vect},$$

where the first arrow is the canonical action of  $\mathrm{QCoh}(-)$  on  $\mathrm{IndCoh}(-)$ .



7.5.3. The discussion in Sect. 7.5.2 applies in particular to  $\mathcal{X} = X_Y^\wedge$ .

Passing to dual functors in (7.9), and using Corollary 7.2.6 we obtain another commutative diagram:

$$(7.11) \quad \begin{array}{ccc} \mathrm{QCoh}(X_Y^\wedge) & \xrightarrow{\widehat{i}_?} & \mathrm{QCoh}(X) \\ \Psi_{X_Y^\wedge} \uparrow & & \uparrow \Psi_X \\ \mathrm{IndCoh}(X_Y^\wedge) & \xrightarrow{\widehat{i}_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(X). \end{array}$$

**Lemma 7.5.4.** *The functor  $\Psi_{X_Y^\wedge}$  is t-exact, when we consider the t-structure on  $\mathrm{IndCoh}(X_Y^\wedge)$  of Sect. 2.4.9 and the inductive t-structure on  $\mathrm{QCoh}(X_Y^\wedge)$  of Sect. 7.3.3.*

*Proof.* The assertion follows from the fact that the functors  $\widehat{i}_*^{\mathrm{IndCoh}}$  and  $\widehat{i}_?$  are t-exact and conservative, and the fact that  $\Psi_X$  is t-exact.  $\square$

7.5.5. Now, consider the functors

$$(7.12) \quad \widehat{i}_? \circ \widehat{i}^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X) \text{ and } \widehat{i}_*^{\mathrm{IndCoh}} \circ \widehat{i}^! : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X).$$

**Lemma 7.5.6.** *The functor  $\Psi_X : \mathrm{IndCoh}(X) \rightarrow \mathrm{QCoh}(X)$  intertwines the functors of (7.12).*

*Proof.* The functors of (7.12) are isomorphic to

$$\mathrm{Cone}(\mathrm{Id} \rightarrow j_* \circ j^*)[-1] \text{ and } \mathrm{Cone}(\mathrm{Id} \rightarrow j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*})[-1],$$

respectively. So, it is enough to show that the functor  $\Psi_X$  intertwines the functors  $j_* \circ j^*$  and  $j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*}$ . However, the latter follows easily from [GL:IndCoh], Propositions 3.1.1 and 3.3.4  $\square$

Combining this with the fact that the horizontal arrows in (7.11) are conservative (in fact, fully faithful), we obtain:

**Corollary 7.5.7.** *The diagram of functors*

$$(7.13) \quad \begin{array}{ccc} \mathrm{QCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^*} & \mathrm{QCoh}(X) \\ \Psi_{X_Y^\wedge} \uparrow & & \uparrow \Psi_X \\ \mathrm{IndCoh}(X_Y^\wedge) & \xleftarrow{\widehat{i}^!} & \mathrm{IndCoh}(X) \end{array}$$

*is commutative.*

Passing to dual functors in (7.13), we obtain yet another commutative diagram of functors:

$$(7.14) \quad \begin{array}{ccc} \mathrm{QCoh}(X_Y^\wedge) & \xrightarrow{\widehat{i}_?} & \mathrm{QCoh}(X) \\ \Psi_{X_Y^\wedge}^\vee \downarrow & & \downarrow \Psi_X^\vee \\ \mathrm{IndCoh}(X_Y^\wedge) & \xrightarrow{\widehat{i}_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(X). \end{array}$$

**7.6. QCoh and IndCoh in the eventually coconnective case.** In this subsection we will assume that  $X$  is eventually coconnective. By Proposition 6.6.6, the ind-scheme  $X_Y^\wedge$  is also eventually coconnective.

7.6.1. Let us recall (see [GL:IndCoh], Proposition 1.4.3) that for  $X$  eventually coconnective, the functor  $\Psi_X : \text{IndCoh}(X) \rightarrow \text{QCoh}(X)$  admits a left adjoint, denoted  $\Xi_X$ . It is characterized by the property that it sends  $\text{QCoh}(X)^{\text{perf}} \simeq \text{QCoh}(X)^c$  to  $\text{Coh}(X) \simeq \text{IndCoh}(X)^c$  via the tautological map

$$\text{QCoh}(X)^{\text{perf}} \rightarrow \text{Coh}(X),$$

which is well-defined because  $X$  is eventually coconnective.

Also, recall that the functor  $\Xi_X^\vee : \text{IndCoh}(X) \rightarrow \text{QCoh}(X)$  is the right adjoint of  $\Psi_X^\vee$ , and it can be described as

$$\Xi_X^\vee \simeq \underline{\text{Hom}}_{\text{QCoh}(X)}(\omega_X, -),$$

(see [GL:IndCoh], Lemma 8.5.4).

We emphasize that the functors  $\Xi$  and  $\Xi^\vee$  are defined specifically for DG schemes, and not arbitrary eventually coconnective objects of  $\text{PreStk}_{\text{laft}}$ .

However, for any object  $\mathcal{Y} \in \text{PreStk}_{\text{laft}}$  we can still ask whether the right adjoint  $\Xi_{\mathcal{Y}}^\vee$  of  $\Psi_{\mathcal{Y}}^\vee$  exists, and if  $\mathcal{Y}$  is equipped with a data of self-duality for  $\text{QCoh}(\mathcal{Y})$  and  $\text{IndCoh}(\mathcal{Y})$ . In this case, the functor  $\Psi_{\mathcal{Y}}$  is well-defined and we can ask whether the left adjoint  $\Xi_{\mathcal{Y}}$  of  $\Psi_{\mathcal{Y}}$  exists.

7.6.2. By passing to right (resp., left) adjoint functors in Diagrams (7.11) and (7.13), we obtain two more commutative diagrams

$$(7.15) \quad \begin{array}{ccc} \text{QCoh}(X_{\mathcal{Y}}^\wedge) & \xleftarrow{\widehat{i}^*} & \text{QCoh}(X) \\ \Xi_{X_{\mathcal{Y}}^\wedge}^\vee \uparrow & & \uparrow \Xi_X^\vee \\ \text{IndCoh}(X_{\mathcal{Y}}^\wedge) & \xleftarrow{\widehat{i}^\dagger} & \text{IndCoh}(X), \end{array}$$

and

$$(7.16) \quad \begin{array}{ccc} \text{QCoh}(X_{\mathcal{Y}}^\wedge) & \xrightarrow{\widehat{i}^\dagger} & \text{QCoh}(X) \\ \Xi_{X_{\mathcal{Y}}^\wedge} \downarrow & & \downarrow \Xi_X \\ \text{IndCoh}(X_{\mathcal{Y}}^\wedge) & \xrightarrow{\widehat{i}_*^{\text{IndCoh}}} & \text{IndCoh}(X). \end{array}$$

In particular, we obtain that the functors  $\Xi_{X_{\mathcal{Y}}^\wedge}^\vee$  and  $\Xi_{X_{\mathcal{Y}}^\wedge}$ , are defined for the DG indscheme  $X_{\mathcal{Y}}^\wedge$ .

7.6.3. We now claim the following:

**Proposition 7.6.4.** *The diagrams of functors*

$$(7.17) \quad \begin{array}{ccc} \text{QCoh}(X_{\mathcal{Y}}^\wedge) & \xleftarrow{\widehat{i}^*} & \text{QCoh}(X) \\ \Xi_{X_{\mathcal{Y}}^\wedge} \downarrow & & \downarrow \Xi_X \\ \text{IndCoh}(X_{\mathcal{Y}}^\wedge) & \xleftarrow{\widehat{i}^\dagger} & \text{IndCoh}(X) \end{array}$$

and

$$(7.18) \quad \begin{array}{ccc} \text{QCoh}(X_{\mathcal{Y}}^\wedge) & \xrightarrow{\widehat{i}^\dagger} & \text{QCoh}(X) \\ \Xi_{X_{\mathcal{Y}}^\wedge}^\vee \uparrow & & \uparrow \Xi_X^\vee \\ \text{IndCoh}(X_{\mathcal{Y}}^\wedge) & \xrightarrow{\widehat{i}_*^{\text{IndCoh}}} & \text{IndCoh}(X). \end{array}$$

are commutative as well.

*Proof.* The two diagrams are obtained from one another by passing to dual functors. Therefore, it is sufficient to show that (7.18) is commutative. Taking into account (7.15) and the fact that in the latter diagram the horizontal arrows are co-localizations, it suffices to show that the functor  $\Xi_X^\vee$  intertwines the functors

$$\widehat{i}_? \circ \widehat{i}^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X) \text{ and } \widehat{i}_*^{\mathrm{IndCoh}} \circ \widehat{i}^! : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X).$$

As in the proof of Lemma 7.5.6, it suffices to show that  $\Xi_X^\vee$  intertwines the functors

$$j_* \circ j^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X) \text{ and } j_*^{\mathrm{IndCoh}} \circ j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X).$$

It is clear that

$$j^* \circ \Xi_X^\vee \simeq \Xi_U^\vee \circ j^{\mathrm{IndCoh},*}.$$

So, we have to show that the natural map

$$\Xi_X^\vee \circ j_*^{\mathrm{IndCoh}} \rightarrow j_* \circ \Xi_U^\vee$$

is an isomorphism. Let  $\mathcal{F}_X \in \mathrm{QCoh}(X)$  and  $\mathcal{F}_U \in \mathrm{IndCoh}(U)$  be two objects. We have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{F}_X, \Xi_X^\vee \circ j_*^{\mathrm{IndCoh}}(\mathcal{F}_U)) &\simeq \mathrm{Hom}_{\mathrm{IndCoh}(X)}(\mathcal{F}_X \otimes \omega_X, j_*^{\mathrm{IndCoh}}(\mathcal{F}_U)) \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{IndCoh}(U)}(j^{\mathrm{IndCoh},*}(\mathcal{F}_X \otimes \omega_X), \mathcal{F}_U) \simeq \mathrm{Hom}_{\mathrm{IndCoh}(U)}(j^*(\mathcal{F}_X) \otimes j^{\mathrm{IndCoh},*}(\omega_X), \mathcal{F}_U) \simeq \\ &\simeq \mathrm{Hom}_{\mathrm{IndCoh}(U)}(j^*(\mathcal{F}_X) \otimes \omega_U, \mathcal{F}_U) \simeq \mathrm{Hom}_{\mathrm{QCoh}(U)}(j^*(\mathcal{F}_X), \Xi_U^\vee(\mathcal{F}_U)). \end{aligned}$$

□

## 8. FORMALLY SMOOTH DG INDSCHEMES

**8.1. The notion of formal smoothness.** Let  $\mathcal{X}_{cl}$  be an object of  ${}^{cl}\mathrm{PreStk}$ .

**Definition 8.1.1.** We say that  $\mathcal{X}_{cl}$  is formally smooth if for every closed nil-immersion

$$S_1 \hookrightarrow S_2$$

of classical affine schemes (i.e., a closed embedding inducing an isomorphism of the underlying reduced schemes), the map of sets

$$\pi_0(\mathcal{X}_{cl}(S_2)) \rightarrow \pi_0(\mathcal{X}_{cl}(S_1))$$

is surjective.

Now, let  $\mathcal{X}$  be an object of  $\mathrm{PreStk}$ .

**Definition 8.1.2.** We say that  $\mathcal{X}$  is formally smooth if:

- (1) The classical prestack  ${}^{cl}\mathcal{X} := \mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$  is formally smooth in the sense of Definition 8.1.1.
- (2) For every  $n$  and  $S \in \mathrm{DGSch}^{\mathrm{aff}}$ , the map  $\mathcal{X}(S) \rightarrow \mathcal{X}(\tau^{\leq n}(S))$  induces an isomorphism on  $\pi_n$ .

We can reformulate Definition 8.1.2 as follows.

**Lemma 8.1.3.** Let  $\mathcal{X} \in \mathrm{PreStk}$  such that  ${}^{cl}\mathcal{X}$  is formally smooth. Then  $\mathcal{X}$  is formally smooth if and only if  $\mathcal{X}$  is convergent and for any integers  $i \geq j$  and  $S \in {}^{\leq i}\mathrm{DGSch}^{\mathrm{aff}}$  the map

$$\mathcal{X}(S) \rightarrow \mathcal{X}(\tau^{\leq j}(S))$$

induces an isomorphism on  $\pi_j$ .

8.1.4. As was alluded to in the introduction, the property of formal smoothness, in both the classical and derived contexts, has a substantial drawback of being non-local in the Zariski topology. For example, we could have given a different definition by requiring the corresponding properties to hold after Zariski localization with respect to the test affine scheme  $S$ . We will see a manifestation of this phenomenon in Sect. 8.2.5 for 0-truncated prestacks that admit connective deformation theory.

However, it will turn out that in the latter case the difference between the two definitions disappears if we restrict ourselves to prestacks locally almost of finite type (see Sect. 8.3), which will be the main case of interest in this paper.

8.1.5. All prestacks that we consider in this paper are 0-truncated in the sense of [GL:Stacks], Sect. 1.1.7. I.e., we consider prestacks  $\mathcal{Y}$  such that for all  $n$  and  $S \in {}^{\leq n}\mathrm{DGSch}^{\mathrm{aff}}$ ,

$$\mathcal{Y}(S) \in n\text{-Grpd} \subset \infty\text{-Grpd}.$$

In this case, we have the following reformulation of the Definition 8.1.2.

**Lemma 8.1.6.** *Let  $\mathcal{X} \in \mathrm{PreStk}$  be a 0-truncated prestack such that  ${}^{cl}\mathcal{X}$  is formally smooth as a classical prestack. Then  $\mathcal{X}$  is formally smooth if and only if for every  $n$  and  $S \in \mathrm{DGSch}^{\mathrm{aff}}$ , the map*

$$\mathcal{X}(S) \rightarrow \mathcal{X}(\tau^{\leq n}(S))$$

*identifies the right-hand side with the  $n$ -truncation of (the Postnikov tower of)  $\mathcal{X}(S)$ .*

8.1.7. For future use, let us note the following:

Let  $\mathcal{X}$  be a DG indscheme, and let  $Y$  be a reduced classical scheme, equipped with a closed embedding  $Y \hookrightarrow {}^{cl,red}\mathcal{X}$ . Consider the formal completion  $\mathcal{X}_Y^\wedge$ .

**Proposition 8.1.8.** *The following conditions are equivalent:*

- (a)  $\mathcal{X}$  is formally smooth.
- (b) For every  $Y \hookrightarrow {}^{cl,red}\mathcal{X}$  as above, the formal completion  $\mathcal{X}_Y^\wedge$  is formally smooth.

*Proof.* Since  $\mathcal{X}_Y^\wedge(S)$  is a connected component of  $\mathcal{X}(S)$ , condition (a) implies condition (b). For the opposite implication, write  ${}^{cl}\mathcal{X}$  as  $\mathrm{colim}_\alpha X_\alpha$ . We claim that it is enough to show that each  $\mathcal{X}_{X_\alpha}^\wedge$  is formally smooth. Indeed, both conditions of formal smoothness can be checked separately over each point of  ${}^{cl,red}S \rightarrow {}^{cl,red}\mathcal{X}$ , and every such point factors through some  $X_\alpha$ .  $\square$

## 8.2. Formal smoothness via deformation theory.

8.2.1. Assume that  $\mathcal{X} \in \mathrm{PreStk}$  admits connective deformation theory, see Definition 4.8.1.

**Proposition 8.2.2.** *Assume that  ${}^{cl}\mathcal{X}$  is 0-truncated. Then  $\mathcal{X}$  is formally smooth if and only if the following equivalent conditions hold:*

- (a) For every  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  and  $x : S \rightarrow \mathcal{X}$ , the object

$$T_x^*\mathcal{X} \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$$

*has the property that*

$$\mathrm{Hom}(T_x^*\mathcal{X}, \mathcal{F}[i]) = 0, \forall \mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit \text{ and } i > 0.$$

- (b) Same as (a), but  $S$  is a classical affine scheme.
- (c) Under an additional assumption that  $\mathcal{X}$  is locally almost of finite type, the same as (b), but  $S$  is assumed reduced.

*Proof.* It is clear that if  $\mathcal{X}$  is formally smooth, then it satisfies (a): indeed, consider the square-zero extension of  $S$  corresponding to  $\mathcal{F}[i]$ . The converse implication follows from deformation theory using Lemma 8.1.3: for every  $i$ , we can view  $\tau^{\leq i+1}(S)$  as a square-zero extension of  $\tau^{\leq i}(S)$ .

Condition (a) implies condition (b) tautologically. The converse implication follows from the fact that every object of  $\mathrm{QCoh}(S)^\heartsuit$  is the direct image under the canonical map  $\tau^{cl}(S) \rightarrow S$ . Indeed, for a point  $x : S \rightarrow \mathcal{X}$ , the pull-back of  $T_x^* \mathcal{X}$  under  $\tau^{cl}(S) \rightarrow S$  identifies with  $T_{cl,x}^* \mathcal{X}$ , where  $^{cl}x$  is the composition  $\tau^{cl}(S) \rightarrow S \xrightarrow{x} \mathcal{X}$ .

Condition (b) implies condition (c) tautologically. For the converse implication, we note that under the assumption that  $\mathcal{X}$  is locally of finite type, by Lemma 5.3.4, the functor  $T_x^* \mathcal{X}$  commutes with colimits in  $\mathrm{QCoh}(S)^\heartsuit$ . This allows to replace any  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$  by one obtained as a direct image from  $^{red}S$ . □

8.2.3. Let  $S$  be an affine classical scheme. Let us characterize those objects

$$F \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\leq 0})$$

that satisfy property (a) of Proposition 8.2.2.

Recall the t-structure on  $\mathrm{Pro}(\mathrm{QCoh}(S))$  introduced in Sect. 4.2.2. We have:

**Lemma 8.2.4.** *For  $S \in \mathrm{Sch}^{\mathrm{aff}}$  and  $F \in \mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$  the following are equivalent:*

- (a)  $H^0(F(\mathcal{F}[i])) = 0$  for all  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$  and  $i \in [1, \dots, n]$ .
- (b)  $H^i(F) = 0$  for  $i \in [-n, -1]$  and  $H^0(F)$ , regarded as a functor  $\mathrm{QCoh}(S)^\heartsuit \rightarrow \mathrm{Vect}^\heartsuit$ , is exact.
- (c)  $H^i(F) = 0$  for  $i \in [-n, -1]$  and  $H^0(F)$  is pro-projective, i.e., belongs to  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}})$  where  $\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}}$  is the subcategory of projective objects in the abelian category  $\mathrm{QCoh}(S)^\heartsuit$ .

*Remark.* This lemma is not specific to  $\mathrm{QCoh}(S)$ ; it is applicable to any stable  $\infty$ -category equipped with a t-structure, such that  $\mathrm{QCoh}(S)^\heartsuit$  has enough projectives and injectives.

*Proof.* The implication (c)  $\Rightarrow$  (b) is immediate. Assume that  $F$  satisfies (b), and let us show that it satisfies (a). By assumption,  $F$  is given as an object

$$\text{“}\lim_{\alpha \in A}\text{” } \mathcal{F}_\alpha \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit),$$

where the category of indices  $A$  is filtered. By definition,

$$H^0(F(\mathcal{F})) \simeq \mathrm{colim}_{\alpha \in A} \mathrm{Hom}(\mathcal{F}_\alpha, \mathcal{F}).$$

Hence, if  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$  is injective, then  $H^0(F(\mathcal{F}[i])) = 0$  for  $i > 0$ . The exactness of  $F$  on the abelian category implies that  $H^0(F(\mathcal{F}[1])) = 0$  for any  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$  by the long exact cohomology sequence. The assertion that  $H^0(F(\mathcal{F}[i])) = 0$  for  $n > i > 1$  and any  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$  follows by induction on  $i$  again by the long exact cohomology sequence.

Finally, assume that  $F$  satisfies (a), and let us deduce (c). Consider the category

$$\{P \in \mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}}, f_P \in H^0(F(P))\}.$$

The assumption implies that this category is cofiltered, and it is easy to see that the resulting map in  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\geq -n, \leq 0})$

$$F \rightarrow \text{“}\lim_{(P, f_P)}\text{” } P$$

is an isomorphism.

□

8.2.5. We can now better explain the non-locality of the definition of formal smoothness mentioned in Sect. 8.1.4:

Let  $S$  be an affine classical scheme, and let  $F$  be an object of  $\mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ . It is a natural question to ask whether the property of  $F$  to belong to  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}})$  is local in the Zariski topology.

Namely, if  $S_i$  is an open cover of  $S$  by affine subchemes and  $F|_{S_i} \in \mathrm{Pro}(\mathrm{QCoh}(S_i)^{\heartsuit, \mathrm{proj}})$ , will it be true that  $F$  itself belongs to  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}})$ ?

Unfortunately, we do not know the answer to this question, but we think that it is probably negative.

*Remark.* Note, however, if we ask the same question for  $F$  being an object of  $\mathrm{QCoh}(S)^\heartsuit$ , rather than  $\mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ , the answer will be positive, due to a non-trivial theorem of Raynaud-Gruson, [RG].

### 8.3. Formal smoothness for prestacks locally of finite type.

8.3.1. Now assume that in the context of Lemma 8.2.4, the object  $F$ , viewed as a functor

$$\mathrm{QCoh}(S)^{\geq -n, \leq 0} \rightarrow \mathrm{Vect}$$

commutes with filtered colimits. In this case we shall say that  $F$  is *pro-coherent*.

Note that this condition is satisfied for  $F$  arising as  $T_x^* \mathcal{X}$ , where  $\mathcal{X} \in {}^{\leq n} \mathrm{PreStk}_{\mathrm{laft}}$ , and  $x : S \rightarrow \mathcal{X}$ .

Also note that when  $S$  is Noetherian, by Lemma 5.3.4, pro-coherence is equivalent to  $F$  belonging to  $\mathrm{Pro}(\mathrm{Coh}(S)^{\geq -n, \leq 0})$ .

In general,  $F$  is pro-coherent if and only if it can be represented by a complex

$$P^{-n-1} \rightarrow P^{-n} \rightarrow \dots \rightarrow P^{-1} \rightarrow P^0$$

in  $\mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$ , whose terms belong to  $\mathrm{Pro}(\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}, \mathrm{f.g.}})$ , where

$$\mathrm{QCoh}(S)^{\heartsuit, \mathrm{proj}, \mathrm{f.g.}} \subset \mathrm{QCoh}(S)^\heartsuit$$

denotes the category of projective finitely generated quasi-coherent sheaves.

8.3.2. We have:

**Lemma 8.3.3.** *For a pro-coherent  $F$ , the property of satisfying the equivalent conditions of Lemma 8.2.4 is local in the Zariski topology.*

*Proof.* First, Lemma 4.2.5 implies that the t-structure on  $\mathrm{Pro}(\mathrm{QCoh}(S))$  is Zariski-local, i.e., an object of  $\mathrm{Pro}(\mathrm{QCoh}(S))$  is connective/coconnective if its restriction to a Zariski cover is connective/coconnective.

Hence, we have to prove that the property of a pro-coherent  $F' \in \mathrm{Pro}(\mathrm{QCoh}(S)^\heartsuit)$  to be pro-projective is Zariski-local. We will show that it is in fact fpqc-local.

Thus, let  $f : S' \rightarrow S$  be an fpqc map, where  $S = \mathrm{Spec}(A)$  and  $S' = \mathrm{Spec}(B)$ . We assume that the functor

$$\mathrm{Pro}(f^*)(F') : (B\text{-mod})^\heartsuit \rightarrow \mathrm{Vect}^\heartsuit$$

is exact, and we wish to deduce the same for  $F' : (A\text{-mod})^\heartsuit \rightarrow \mathrm{Vect}^\heartsuit$ . Note that by adjunction,  $\mathrm{Pro}(f^*)(F')(\mathcal{M}) = F'(f_*(\mathcal{M}))$  for  $\mathcal{M} \in (B\text{-mod})^\heartsuit$ .

Consider the object  $F'(A) \in (A\text{-mod})^\heartsuit$ . There is a natural map of functors  $(A\text{-mod})^\heartsuit \rightarrow \text{Vect}^\heartsuit$

$$(8.1) \quad \mathcal{N} \otimes_A F'(A) \rightarrow F'(\mathcal{N}),$$

where in the above formula we are using the *non-derived* tensor product.

Note the map in (8.1) is an isomorphism whenever  $F'$  is pro-coherent and pro-projective. Indeed, both functors are right exact and commute with filtered colimits, so the isomorphism for any  $\mathcal{N}$  follows from the case  $\mathcal{N} = A$ .

And vice versa, if (8.1) is an isomorphism then  $F'$  is pro-projective. Indeed, the left-hand side is a right exact, and the right-hand side is left exact, so if the map in question is an isomorphism, both functors are actually exact.

Also note that (8.1) is an isomorphism for  $F'$  pro-coherent whenever  $\mathcal{N}$  is  $A$ -flat, by Lazard's lemma.

In order to show that (8.1) is an isomorphism under our assumption, let us tensor both sides with  $B$ , and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{N} \otimes_A F'(A) \otimes_A B & \longrightarrow & F'(\mathcal{N}) \otimes_A B \\ \downarrow & & \downarrow \\ \mathcal{N} \otimes_A F'(B) & \longrightarrow & F'(\mathcal{N} \otimes_A B). \end{array}$$

Since  $B$  is faithfully flat over  $A$ , it is enough to show that the upper horizontal arrow is an isomorphism.

In the above diagram the vertical arrows are isomorphisms since  $B$  is  $A$ -flat. However, the lower horizontal arrow identifies with

$$\mathcal{N} \otimes_A F'(B) \simeq (\mathcal{N} \otimes_A B) \otimes_B F'(B) \rightarrow F'(\mathcal{N} \otimes_A B) \simeq \text{Pro}(f^*)(F')(\mathcal{N} \otimes_A B),$$

which is an isomorphism by (8.1) applied to  $\text{Pro}(f^*)(F')$ . □

8.3.4. In view of Proposition 8.2.2, the above lemma implies that for 0-truncated prestacks that admit connective deformation theory and are locally almost of finite type, the definition of formal smoothness is reasonable, in the sense that it is Zariski-local.

As a manifestation of this, we will establish the following assertion that will be useful in the sequel.

8.3.5. Let  $\mathcal{X}$  be a formal DG scheme with the underlying reduced classical scheme  $X$ . Denote

$$T^*\mathcal{X}|_X := T_x^*\mathcal{X},$$

where  $x : X \rightarrow \mathcal{X}$  is the tautological point.

**Corollary 8.3.6.** *Suppose that  $\mathcal{X}$  is locally almost of finite type, and that  $T^*\mathcal{X}|_X$  locally satisfies the equivalent conditions of Lemma 8.2.4 for any  $n$ . Then  $\mathcal{X}$  is formally smooth.*

*Proof.* We will check that the conditions of Proposition 8.2.2(c). Note that every map  $S \rightarrow \mathcal{X}$ , where  $S$  is a reduced classical affine scheme, factors through a map  $S \rightarrow X$ .

Note that by Proposition 5.3.2, the locally almost of finite type assumption on  $\mathcal{X}$  implies that  $T^*\mathcal{X}|_X \in \text{Pro}(\text{Coh}(X)^{\leq 0})$ . Thus, we have to check that if  $F \in \text{Pro}(\text{Coh}(X)^{\leq 0})$  locally

satisfies conditions of Lemma 8.2.4, then so does its pull-back under any map. However, the latter follows from Lemma 8.3.3.  $\square$

**8.4. Examples of formally smooth DG ind schemes.** In this subsection we will give three examples of formally smooth DG ind schemes.

8.4.1. The first example is the most basic one: we claim that the affine space  $\mathbb{A}^n$ , considered as an object of  $\text{PreStk}$ , is formally smooth. Indeed, the definition of formal smoothness is satisfied on the nose as

$$\text{Maps}(\text{Spec}(A), \mathbb{A}^n) \simeq \Omega^\infty(\text{Sp}(A))^{\times n}.$$

8.4.2. Let  $X$  be a classical smooth scheme of finite type over  $k$ . We claim that  $X$ , considered as an object of  $\text{PreStk}$  (i.e.,  ${}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(X)$ ), is formally smooth.

Indeed, it suffices to show that the conditions of Proposition 8.2.2 are satisfied. In fact, we claim that  $T^*X$ , is an object of  $\text{Coh}(X)^\heartsuit$ , and is locally projective.

The question is local on  $X$ , so we can assume that  $X$  fits into a Cartesian square

$$(8.2) \quad \begin{array}{ccc} X & \longrightarrow & \mathbb{A}^n \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & \mathbb{A}^m, \end{array}$$

where the map  $f$  is smooth, and where the Cartesian product is taken in the category of classical schemes.

Since  $f$  is flat, the above square is also Cartesian in the category of DG schemes. Hence,  $T^*X$  can be calculated as

$$\text{Cone}(f^*(T^*\mathbb{A}^m)|_X \rightarrow T^*\mathbb{A}^n|_X),$$

and the smoothness hypothesis on  $f$  implies the required properties of  $T^*X$ .

**Corollary 8.4.3.** *Let  $X$  be a smooth classical scheme locally of finite type, and let  $Y \subset X$  be a Zariski-closed subset. Then the formal completion  $X_Y^\wedge$  is formally smooth as an object of  $\text{PreStk}$ .*

*Proof.* This follows from Proposition 8.2.2 as  $\widehat{i} : X_Y^\wedge \rightarrow X$  induces an isomorphism on cotangent complexes.  $\square$

Also, note that by Proposition 6.7.2, the DG ind scheme  $X_Y^\wedge$  is 0-coconnective, i.e., is a left Kan extension of a classical ind scheme.

8.4.4. The following example will be needed for the proof of Theorem 9.1.2. Consider the formal DG scheme  $\mathbb{A}^{n,m} := \text{Spf}(k[x_1, \dots, x_n][[y_1, \dots, y_m]])$ , i.e., the formal completion of  $\mathbb{A}^{n+m}$  along the subscheme  $\mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$  embedded along the first  $n$  coordinates.

Let now  $f_1, \dots, f_k$  be elements of  $k[x_1, \dots, x_n][[y_1, \dots, y_m]]$ , and let  $\bar{f}_1, \dots, \bar{f}_k$  be their images under

$$k[x_1, \dots, x_n][[y_1, \dots, y_m]] \twoheadrightarrow k[x_1, \dots, x_n].$$

Set

$$\mathcal{X} := 0 \times_{\mathbb{A}^k} \mathbb{A}^{n,m} \text{ and } X := 0 \times_{\mathbb{A}^k} \mathbb{A}^n.$$

Assume now that the Jacobi matrix of  $f_1, \dots, f_k$  is non-degenerate when restricted to  $X$ . I.e., the matrix  $k \times (m+n)$ -matrix  $\partial_i(f_j)|_X$ , viewed as a map

$$\mathcal{O}_X^{\oplus n+m} \rightarrow \mathcal{O}_X^{\oplus k},$$



is a surjective map of vector bundles when restricted to  $X$ .

From Corollary 8.3.6 and Corollary 6.3.2, we obtain:

**Corollary 8.4.5.** *Under the above circumstances, the DG indscheme  $\mathcal{X}$  is formally smooth.*

We now claim:

**Proposition 8.4.6.** *The DG indscheme  $\mathcal{X}$  is 0-coconnective.*

*Proof.* Consider the scheme

$$\sim\mathbb{A}^{n,m} := \operatorname{Spec}(k[x_1, \dots, x_n][y_1, \dots, y_m])$$

and its map to  $\mathbb{A}^k$  given by  $f_1, \dots, f_k$ . The assumption on the Jacobi matrix implies that this map is flat on a Zariski neighborhood  $U$  of  $X \subset \sim\mathbb{A}^{n,m}$ . Therefore, the Cartesian product *taken in the category of DG schemes*

$$\sim\mathcal{X} \simeq 0 \times_{\mathbb{A}^k} U$$

is 0-coconnective as a DG scheme.

The formal DG scheme  $\mathcal{X}$  is obtained from  $\sim\mathcal{X}$  as a formal completion along  $X$ . Since all the schemes involved are Noetherian, the assertion follows from Proposition 6.7.2.  $\square$

## 9. CLASSICAL VS. DERIVED FORMAL SMOOTHNESS

The focus of this section is the relation between the notions of formal smoothness in the classical and derived contexts when  $\mathcal{X}$  is a DG indscheme. Namely, we would like to know under what circumstances a formally smooth DG indscheme  $\mathcal{X}$  is 0-coconnective, i.e., arises as a left Kan extension from a classical indscheme. The reader may have observed that this was the case in all the examples that we considered in Sect. 8.4.

And vice versa, we would like to know when, for a classical formally smooth indscheme  $\mathcal{X}_{cl}$ , the object

$$\mathcal{X} := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\operatorname{aff}} \hookrightarrow \operatorname{DGSch}^{\operatorname{aff}}}(\mathcal{X}_{cl}) \in \operatorname{Stk}$$

is a formally smooth DG indscheme. (Note that it is not clear that  $\mathcal{X}$  defined as above is a DG indscheme, since the convergence condition is not a priori guaranteed.)

Unfortunately, we do not have a general answer for this question even in the case of schemes: we do not even know that the DG scheme

$$X := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\operatorname{aff}} \hookrightarrow \operatorname{DGSch}^{\operatorname{aff}}}(X_{cl})$$

is smooth when  $X_{cl}$  is a smooth classical scheme, except when  $X_{cl}$  is locally of finite type.

**9.1. The main result.** The main result of this section and the first of the two main results of this paper is a partial answer to the above questions, under the assumption that our (DG) indschemes are locally (almost) of finite type.

**9.1.1.** Let  $\mathcal{X}_{cl}$  be a classical formally smooth  $\aleph_0$  indscheme. Assume that  $\mathcal{X}_{cl}$  is locally of finite type. Set  $\mathcal{X} := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\operatorname{aff}} \hookrightarrow \operatorname{DGSch}^{\operatorname{aff}}}(\mathcal{X}_{cl})$ .

We will prove:

**Theorem 9.1.2.** *Under the above circumstances,  $\mathcal{X}$  is a formally smooth DG indscheme.*

This theorem gives a partial answer to the second of the two questions above. We shall presently show that it also gives a partial answer to the first question.

9.1.3. Let us make the following observation:

**Proposition 9.1.4.** *If  $\mathcal{X}$  is a formally smooth DG indscheme such that*

$${}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{cl}\mathcal{X}) \in \mathrm{Stk}$$

*is also a formally smooth DG indscheme, then the natural map*

$${}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{cl}\mathcal{X}) \rightarrow \mathcal{X}$$

*is an isomorphism. In particular,  $\mathcal{X}$  is 0-coconnective.*

*Proof.* By assumption, both sides in

$$(9.1) \quad \mathcal{X}' := {}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}({}^{cl}\mathcal{X}) \rightarrow \mathcal{X}$$

are formally smooth DG indschemes, and the above map induces an isomorphism of the underlying classical indschemes.

By deformation theory, it suffices to show that for every affine DG scheme  $S$  and a map  $x' : S \rightarrow \mathcal{X}'$ , the map

$$T_x^*\mathcal{X} \rightarrow T_{x'}^*\mathcal{X}'$$

is an isomorphism, where  $x$  is the composition of  $x'$  and the map (9.1).

Using Proposition 8.2.2(a), we obtain that it suffices to check that the map

$$(9.2) \quad T_{x'}^*\mathcal{X}'(\mathcal{F}) \rightarrow T_x^*\mathcal{X}(\mathcal{F})$$

is an isomorphism for every  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$ . Since any such  $\mathcal{F}$  comes as a direct image from  $\tau^{cl}(S)$ , this reduces the assertion to the case when  $S$  is classical.

We have

$$T_{x'}^*\mathcal{X}'(\mathcal{F}) \simeq \mathrm{Maps}(S_{\mathcal{F}}, \mathcal{X}') \times_{\mathrm{Maps}(S, \mathcal{X}')} x',$$

and similarly for  $T_x^*\mathcal{X}(\mathcal{F})$ . When  $S$  is classical and  $\mathcal{F} \in \mathrm{QCoh}(S)^\heartsuit$ , the DG scheme  $S_{\mathcal{F}}$  is also classical. So, both sides of (9.2) only depend on the restrictions of  $\mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$  and  $\mathcal{X}'|_{\mathrm{Sch}^{\mathrm{aff}}}$ , respectively, and, hence are isomorphic by construction.  $\square$

9.1.5. Combining Proposition 9.1.4 and Theorem 9.1.2, we obtain:

**Theorem 9.1.6.** *Let  $\mathcal{X}$  be a formally smooth DG indscheme, such that  ${}^{cl}\mathcal{X} := \mathcal{X}|_{\mathrm{Sch}^{\mathrm{aff}}}$  is locally of finite type and  $\aleph_0$ . Then  $\mathcal{X}$  is 0-coconnective. Moreover,  $\mathcal{X}$  is locally almost of finite type and weakly  $\aleph_0$ .*

*Proof.* The first assertion is immediate.

The fact that  $\mathcal{X}$  is locally almost of finite type follows from the fact that for  $n' \leq n$ , the functor

$${}^{n'}L\mathrm{LKE}_{\leq n' \mathrm{DGSch}^{\mathrm{aff}} \hookrightarrow \leq n \mathrm{DGSch}^{\mathrm{aff}}} : \leq n' \mathrm{Stk} \rightarrow \leq n \mathrm{Stk}$$

sends *truncated* objects of  $\leq n' \mathrm{Stk}_{\mathrm{lft}}$  to  $\leq n \mathrm{Stk}_{\mathrm{lft}}$ , which follows from [GL:Stacks], Proposition 2.5.6.

The fact that  $\mathcal{X}$  is weakly  $\aleph_0$  follows from Proposition 5.2.3.  $\square$

Thus, we obtain:

**Corollary 9.1.7.** *There exists an equivalence of categories between the category of classical formally smooth  $\aleph_0$  indschemes locally of finite type and that of formally smooth weakly  $\aleph_0$  DG indschemes locally almost of finite type.*

9.1.8. Prior to proving Theorem 9.1.2, let us see some of its corollaries in concrete geometric situations.

## 9.2. Loop spaces.

9.2.1. Let  $Z$  be an object of  $\text{PreStk}$ . We define the objects  $Z[t]/t^k$ ,  $Z[[t]]$  and  $Z((t))$  of  $\text{PreStk}$  as follows: for  $S = \text{Spec}(A) \in \text{DGSch}^{\text{aff}}$ ,

$$\text{Hom}(S, Z[t]/t^k) := \text{Hom}(\text{Spec}(A[t]/t^k), Z), \quad \text{Hom}(S, Z[[t]]) := \text{Hom}(\text{Spec}(A[[t]]), Z)$$

and

$$\text{Hom}(S, Z((t))) := \text{Hom}(\text{Spec}(A((t))), Z).$$

Note that by definition,

$$Z[[t]] \simeq \varprojlim_k Z[t]/t^k,$$

as objects of  $\text{PreStk}$ .

**Lemma 9.2.2.** *Assume that  $Z$  is formally smooth as an object of  $\text{PreStk}$ . Then so are  $Z[t]/t^k$ ,  $Z[[t]]$  and  $Z((t))$ .*

*Proof.* This is immediate from the fact that for a DG algebra  $A$ , the maps

$$\tau^{\leq n}(A[t]/t^k) \rightarrow (\tau^{\leq n}(A))[t]/t^k, \quad \tau^{\leq n}(A[[t]]) \rightarrow (\tau^{\leq n}A)[[t]] \quad \text{and} \quad \tau^{\leq n}(A((t))) \rightarrow (\tau^{\leq n}A)((t))$$

are isomorphisms, and that for a surjection of classical algebras  $A_1 \rightarrow A_2$ , the corresponding maps

$$A_1[t]/t^k \rightarrow A_2[t]/t^k, \quad A_1[[t]] \rightarrow A_2[[t]] \quad \text{and} \quad A_1((t)) \rightarrow A_2((t))$$

are also surjections. □

9.2.3. From now on we are going to consider the case when  $Z \in \text{DGSch}_{\text{laft}}$ . We have:

**Proposition 9.2.4.** *Under the above circumstances, we have:*

- (a)  $Z[t]/t^k \in \text{DGSch}_{\text{laft}}$ , and is affine if  $Z$  is affine.
- (b)  $Z[[t]] \in \text{DGSch}$ , and it is affine if  $Z$  is affine.
- (c) If  $Z$  is affine, then  $Z((t))$  is a DG indscheme.

*Proof.* For all three statements, it is enough to assume that  $Z$  is affine. Note that  $Z[t]/t^k$ ,  $Z[[t]]$  and  $Z((t))$ , considered as objects of  $\text{PreStk}$  are convergent. Hence, it is sufficient to show that

$$\begin{aligned} \leq^n(Z[t]/t^k) &:= Z[t]/t^k|_{\leq^n \text{DGSch}^{\text{aff}}}, \quad \leq^n(Z[[t]]) := Z[[t]]|_{\leq^n \text{DGSch}^{\text{aff}}} \quad \text{and} \\ \leq^n(Z((t))) &:= Z((t))|_{\leq^n \text{DGSch}^{\text{aff}}} \end{aligned}$$

are representable by objects from  $\leq^n \text{DGSch}^{\text{aff}}$  (for the first two) and  $\leq^n \text{DGindSch}$ , respectively. Note that the above objects only depend on the truncation  $\leq^n Z$ . The assertions of the proposition result from combining the following observations:

- (i) The assignments  $Z \mapsto Z[t]/t^k$ ,  $Z \mapsto Z[[t]]$  and  $Z \mapsto Z((t))$  commute with limits.
- (ii) Every object of  $\leq^n \text{DGSch}^{\text{aff}}$  can be obtained as the totalization of a truncated cosimplicial object whose terms are isomorphic to affine spaces  $\mathbb{A}^n$ .
- (iii) The subcategories

$$\leq^n \text{DGSch}^{\text{aff}} \subset \leq^n \text{PreStk} \quad \text{and} \quad \leq^n \text{DGindSch} \subset \leq^n \text{PreStk}$$

are stable under finite products.

- (iv) For  $Z = \mathbb{A}^n$ , both assertions of the proposition are manifest. □

9.2.5. Suppose now that  $Z$  is a classical scheme which is smooth over  $k$  (and in particular, locally of finite type). We have:

**Proposition 9.2.6.** *The DG schemes  $Z[t]/t^k$  and  $Z[[t]]$  are 0-coconnective.*

*Proof.* To prove that  $Z[t]/t^k$  is 0-coconnective, by Proposition 9.1.4 and Sect. 8.4.2, it is sufficient that the classical scheme

$${}^{cl}(Z[t]/t^k) := Z[t]/t^k|_{\text{Sch}^{\text{aff}}}$$

is smooth. By Lemma 9.2.2,  ${}^{cl}(Z[t]/t^k)$  is formally smooth as a classical scheme, which implies that it is smooth, since  ${}^{cl}(Z[t]/t^k)$  is locally of finite type by Proposition 9.2.4(a).

To treat the case of  $Z[[t]]$ , we will have to go back to the proof of Proposition 9.2.4. We can assume that  $Z$  is affine and that it fits into a Cartesian square (8.2). Hence, we have a Cartesian square

$$\begin{array}{ccc} Z[[t]] & \longrightarrow & \mathbb{A}^n[[t]] \\ \downarrow & & \downarrow f[[t]] \\ 0 & \longrightarrow & \mathbb{A}^m[[t]]. \end{array}$$

Since the affine schemes  $\mathbb{A}^n[[t]]$  and  $\mathbb{A}^m[[t]]$  are 0-coconnective, to show that  $Z[[t]]$  is also 0-coconnective, it suffices to show that the map  $f[[t]]$  is flat. The latter is the limit of the maps  $f[t]/t^k : \mathbb{A}^n[t]/t^k \rightarrow \mathbb{A}^m[t]/t^k$ , and smoothness of  $f$  implies that each of these maps is flat. Hence,  $f[t]/t^k$  is flat as well.  $\square$

9.2.7. *Question.* What are the conditions on a classical scheme of finite type  $Z$  (viewed as a 0-coconnective DG scheme), that will guarantee that  $Z[[t]]$  will also be 0-coconnective.

It is easy to see that this is not always the case: for instance, consider  $Z = \text{Spec}(k[t]/t^2)$ . However, the smoothness condition on  $Z$  is not necessary, as can be seen from the following example:

Let  $\mathfrak{g}$  be a semi-simple Lie algebra, and let  $\mathcal{N} \subset \mathfrak{g}$  be its nilpotent cone. We have:

**Corollary 9.2.8.** *The DG scheme  $\mathcal{N}[[t]]$  is 0-coconnective.*

*Proof.* By definition,  $\mathcal{N}$  fits into a Cartesian square

$$\begin{array}{ccc} \mathcal{N} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \varpi \\ 0 & \longrightarrow & \mathfrak{g} // G, \end{array}$$

taken in the category of classical schemes, where  $\mathfrak{g} // G$  is the GIT quotient of  $\mathfrak{g}$  by the adjoint action of  $G$ , i.e.,  $\text{Spec}(\text{Sym}(\mathfrak{g}^*)^G)$ , and  $\varpi$  is the Chevalley map.

However, by Kostant's theorem, the map  $\varpi$  is flat, so the above square is also Cartesian in the category of DG schemes. Hence, we have a Cartesian square

$$\begin{array}{ccc} \mathcal{N}[[t]] & \longrightarrow & \mathfrak{g}[[t]] \\ \downarrow & & \downarrow \varpi[[t]] \\ 0 & \longrightarrow & \mathfrak{g} // G[[t]]. \end{array}$$

Since  $\mathfrak{g}$  and  $\mathfrak{g}/G$  are smooth schemes of finite type (in fact, isomorphic to affine spaces), the DG schemes  $\mathfrak{g}[[t]]$  and  $\mathfrak{g}/G[[t]]$  are 0-coconnective. Hence, to show that  $N[[t]]$  is 0-coconnective, it suffices to know that the map  $\varpi[[t]]$  is flat. However, the latter is Theorem A.4 in [EF].  $\square$

9.2.9. *The case of loops.* Let  $Z$  be an affine smooth scheme of finite type over the ground field. We propose:

**Conjecture 9.2.10.** *The DG indscheme  $Z((t))$  is 0-coconnective.*

In this next subsection we will prove this conjecture in a particular case when  $Z$  is an algebraic group  $G$ .

### 9.3. Loop groups and the affine Grassmannian.

9.3.1. Let  $G$  be an algebraic group. We define the affine Grassmannian  $\mathrm{Gr}_G$  as an object of  $\mathrm{PreStk}$  as follows:

$\mathrm{Maps}(\mathrm{Spec}(A), \mathrm{Gr}_G)$  is the  $\infty$ -groupoid of principal  $G$ -bundles on  $\mathrm{Spec}(A[[t]])$  equipped with a trivialization over  $\mathrm{Spec}(A((t)))$ .

It is easy to show that  $\mathrm{Gr}_G$  is convergent and that it belongs to  $\mathrm{Stk}$  (i.e., it satisfies fpppf descent). We have a naturally defined map  $G((t)) \rightarrow \mathrm{Gr}_G$ , which identifies  $\mathrm{Gr}_G$  with the quotient of  $G((t))$  by  $G[[t]]$  in the fpppf and the étale topology (indeed, it is easy to see that any  $G$ -bundle on  $\mathrm{Spec}(A[[t]])$  admits a trivialization after an étale localization with respect to  $\mathrm{Spec}(A)$ ).

It is well-known that the underlying object  ${}^{cl}\mathrm{Gr}_G \in {}^{cl}\mathrm{Stk}$  is a classical indscheme, which is  $\mathbb{N}_0$  and locally of finite type.

**Proposition 9.3.2.**  *$\mathrm{Gr}_G$  is a DG indscheme. Moreover, it is formally smooth.*

*Proof.* To prove that  $\mathrm{Gr}_G$  is a DG indscheme, we will apply Theorem 5.1.1. For  $S \in \mathrm{DGSch}^{\mathrm{aff}}$  and a point  $g : S \rightarrow \mathrm{Gr}_G$  we need to study the category of extensions of  $g$  to a point  $g' : S' \rightarrow \mathrm{Gr}_G$  for square-zero extensions  $S \hookrightarrow S'$ . The question is local in the étale topology on  $S$ , so we can assume that the point  $g$  admits a lift to a point  $\tilde{g} : S \rightarrow G((t))$ . Multiplication by  $\tilde{g}$  defines a map

$$\mathrm{SplitSqZExt}(S, 1_{G[[t]]}) \rightarrow \mathrm{SplitSqZExt}(S, \tilde{g}),$$

where  $1_{G[[t]]} : S \rightarrow G[[t]]$  is the constant map to the unit point of  $G[[t]]$ . Consider the corresponding map

$$\alpha : T_{\tilde{g}}^* G((t)) \rightarrow T_{1_{G[[t]]}}^* G[[t]].$$

We claim that  $\mathrm{Cone}(\alpha)[-1]$  represents  $T_g^* \mathrm{Gr}_G$ . This follows from the fact that any extension  $g' : S' \rightarrow \mathrm{Gr}_G$  also admits a lift to  $\tilde{g}' : S' \rightarrow G((t))$  and if  $\mathcal{F} \in \mathrm{QCoh}(S)$  is the ideal of  $S$  inside  $S'$ , the ambiguity for such lift is given by the fiber of  $\mathrm{SplitSqZExt}(S, 1_{G[[t]]})$  over  $S_{\mathcal{F}} \in \mathrm{SplitSqZExt}(S)$ . This shows that  $\mathrm{Gr}_G$  satisfies scheme-like Conditions (A) and (C), while Condition (B) follows from the construction.

To show that  $\mathrm{Gr}_G$  is formally smooth, it suffices to show that  $\mathrm{Cone}(\alpha)[-1]$  satisfies property (b) of Proposition 8.2.2. Multiplication by the inverse  $\tilde{g}$  defines an isomorphism between  $\mathrm{Cone}(\alpha)$  and the situation when  $\tilde{g} = 1_{G((t))}$ . In the latter case,  $\mathrm{Cone}(\alpha)$  is isomorphic to the object of  $\mathrm{Pro}(\mathrm{Coh}(S)^{\vee})$  equal to “ $\lim_{\alpha}$ ”  $\mathcal{O}_S \otimes V_{\alpha}^*$ , where  $\alpha \mapsto V_{\alpha}$  is the filtered family of finite-dimensional  $k$ -vector spaces, such that

$$\mathrm{colim}_{\alpha} V_{\alpha} \simeq \mathfrak{g}((t))/\mathfrak{g}[[t]].$$

$\square$

9.3.3. Let us now observe the following corollary of Theorem 9.1.6:

**Theorem 9.3.4.**  *$\mathrm{Gr}_G$  is 0-coconnective. Moreover, it is weakly  $\aleph_0$ , and locally almost of finite type.*

We shall now use Theorem 9.3.4 to prove the following:

**Theorem 9.3.5.** *The indscheme  $G((t))$  is 0-coconnective.*

*Proof.* Let  $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  be a map in  $\mathrm{Stk}$  such that  $\mathcal{X}_2$  is 0-coconnective, and for any  $S \in \mathrm{Sch}^{\mathrm{aff}}$  and a map  $S \rightarrow \mathcal{X}_2$ , the Cartesian product  $S \times_{\mathcal{X}_1} \mathcal{X}_2 \in \mathrm{Stk}$  is also 0-coconnective.

**Lemma 9.3.6.** *Under the above circumstances,  $\mathcal{X}_2$  is also 0-coconnective.*

We apply this lemma to  $\mathcal{X}_1 = G((t))$  and  $\mathcal{X}_2 = \mathrm{Gr}_G$ . It remains to verify that for a classical affine scheme  $S$  and a map  $g : S \rightarrow \mathrm{Gr}_G$ , the Cartesian product  $S \times_{\mathrm{Gr}_G} G((t))$  is 0-coconnective. The question is local in the étale topology on  $S$ . Hence, we can assume that  $g$  admits a lift to an  $S$ -point of  $G((t))$ . However, this lift defines an isomorphism

$$S \times_{\mathrm{Gr}_G} G((t)) \simeq S \times G[[t]],$$

and the assertion follows from Theorem 9.3.4 and Proposition 9.2.6.  $\square$

**9.4. Cotangent complex of a classical formally smooth (ind)scheme.** For the proof of Theorem 9.1.2 we will need to establish several facts concerning the cotangent complex of classical formally smooth ind schemes.

9.4.1. Let  $\mathcal{X}_{cl}$  be a classical indscheme; set

$$\mathcal{X} := {}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{X}_{cl}) \in \mathrm{Stk}.$$

**Proposition 9.4.2.** *The indscheme  $\mathcal{X}_{cl}$  is classically formally smooth if and only if for every  $S \in \mathrm{Sch}^{\mathrm{aff}}$  and  $x : S \rightarrow \mathcal{X}_{cl}$ , the object*

$$\geq^{-1}(T_x^* \mathcal{X}) \in \mathrm{Pro}(\mathrm{QCoh}(S))^{\geq -1, \leq 0}$$

*satisfies the equivalent conditions of Lemma 8.2.4.*

*Proof.* The fact that condition (a) of Lemma 8.2.4 implies the classical formal smoothness of  $\mathcal{X}_{cl}$  follows from Sect. 4.7.4.

For the opposite implication, let us assume that  $\mathcal{X}_{cl}$  is formally smooth. We will check that  $\geq^{-1}(T_x^* \mathcal{X})$  satisfies condition (b) of Lemma 8.2.4.

The fact that  $H^0(T_x^* \mathcal{X})$  is exact follows from the definition of  $T_x^* \mathcal{X}$  in terms of split square-zero extensions in Sect. 4.1.2. It remains to show that  $H^{-1}(T_x^* \mathcal{X}) = 0$ .

Let

$$\mathcal{X}_{cl} \simeq \mathrm{colim}_{\alpha \in A} X_{\alpha},$$

where  $X_{\alpha} \in \mathrm{Sch}_{\mathrm{qc-qs}}$ . Let  $\alpha_0$  be an index such that  $x$  factors through a map  $x_{\alpha_0} : S \rightarrow X_{\alpha_0}$ .

Since the t-structure on  $\mathrm{Pro}(\mathrm{QCoh}(S))$  is Zariski-local, we can assume that the map  $x_{\alpha_0}$  factors as

$$S \rightarrow U_{\alpha_0} \xrightarrow{j} X_{\alpha_0},$$

where  $U_{\alpha_0}$  is an open affine inside  $X_{\alpha_0}$ .

Let  $\iota_{\alpha_0} : X_{\alpha_0} \rightarrow \mathcal{X}_{cl}$  denote the tautological map. For  $(\alpha_0 \rightarrow \alpha) \in \mathbf{A}$ , let  $\iota_{\alpha_0, \alpha}$  denote the corresponding closed embedding  $X_{\alpha_0} \rightarrow X_\alpha$ .

It is easy to see that it is sufficient to show that

$$H^{-1} \left( T_{\iota_{\alpha_0} \circ j}^* \mathcal{X} \right) = 0$$

as an object of  $\text{Pro}(\text{QCoh}(U_{\alpha_0})^\heartsuit)$ .

We have

$$H^{-1} \left( T_{\iota_{\alpha_0} \circ j}^* \mathcal{X} \right) \simeq \varinjlim_{\alpha \in \mathbf{A}_{\alpha_0/}} H^{-1} \left( T_{\iota_{\alpha_0, \alpha} \circ j}^* X_\beta \right).$$

So, we need to show that for a given  $\mathcal{M} \in \text{QCoh}(U_{\alpha_0})^\heartsuit$ ,  $\alpha_0 \rightarrow \alpha$  and

$$\phi : H^{-1}(T_{\iota_{\alpha_0, \alpha} \circ j}^* X_\alpha) \rightarrow \mathcal{M},$$

there exists  $\alpha \rightarrow \beta$ , such that the composition

$$H^{-1}(T_{\iota_{\alpha_0, \beta} \circ j}^* X_\beta) \rightarrow H^{-1}(T_{\iota_{\alpha_0, \alpha} \circ j}^* X_\alpha) \xrightarrow{\phi} \mathcal{M}$$

vanishes.

Since  $U_{\alpha_0}$  is affine and in particular quasi-compact, we can shrink it further and thus assume that it is the pre-image of an open affine  $U_\alpha$  in  $X_\alpha$ . Replacing  $\mathcal{M}$  by its direct image under  $U_{\alpha_0} \rightarrow U_\alpha$ , we can assume that  $\alpha = \alpha_0$ . Further, embedding  $\mathcal{M}$  into an injective sheaf, we can assume that the map  $\phi$  extends to a map  $\psi : T^*U_\alpha \rightarrow \mathcal{M}[1]$ . We wish to find an index  $\beta \in \mathbf{A}_{\alpha/}$  such that the composition

$$(9.3) \quad T_{\iota_{\alpha, \beta} \circ j}^* X_\beta \rightarrow T^*U_\alpha \xrightarrow{\psi} \mathcal{M}[1]$$

vanishes.

However, the data of  $\psi$  as above is equivalent to that of a square-zero extension  $U'_\alpha$  of  $U_\alpha$ . And the data of a splitting of (9.3) is equivalent to that of an extension of the map  $\iota_{\alpha, \beta} \circ j : U_\alpha \rightarrow X_\beta$  to a map  $U'_\alpha \rightarrow X_\beta$ . Thus, giving such an index  $\beta$  is equivalent to extending the map  $U_\alpha \rightarrow \mathcal{X}_{cl}$  to a map  $U'_\alpha \rightarrow \mathcal{X}_{cl}$ . The existence of such an extension follows from the classical formal smoothness of  $\mathcal{X}_{cl}$ . □

**Corollary 9.4.3.** *Let  $X_{cl}$  be a classical scheme, and set*

$$X := {}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(X_{cl}).$$

*Then  $X_{cl}$  is classically formally smooth if and only if  $T^*X$  satisfies:*

- (a)  $H^{-1}(T^*X) = 0$ .
- (b)  $H^0(T^*X)$  is projective over every affine subscheme of  $X$ .

*Proof.* We only need to show that if  $S$  is an affine scheme mapping to  $X$ , then the pull-back of  $T^*X$  to it is projective. By assumption, we know this locally in the Zariski topology on  $S$ . The assertion now follows from the theorem of Raynaud-Gruson mentioned earlier that projectivity of a module is a Zariski-local property. □

9.4.4. The following somewhat technical assertion will be needed in the sequel:

Let  $\mathcal{X}_{cl}$  be a classical formal scheme with the underlying reduced scheme  $X$ , and let  $\mathcal{X}$  be as above.

**Corollary 9.4.5.** *Suppose that  $\mathcal{X}_{cl}$  is locally of finite type, and that  $\geq^{-1}(T^*\mathcal{X}|_X)$  locally satisfies the equivalent conditions of Lemma 8.2.4. Then  $\mathcal{X}_{cl}$  is classically formally smooth.*

This follows from Proposition 9.4.2 in the same way as Corollary 8.3.6 follows from Proposition 8.2.2.

**9.5. Classical formally smooth indschemes locally of finite type case.** In this subsection we will reduce Theorem 9.1.2 to a key proposition (Proposition 9.5.2) that describes the general shape of formal classical schemes locally of finite type.

9.5.1. Let  $\mathcal{X}_{cl}$  be as in Theorem 9.1.2. Let  $Y$  be a reduced classical scheme and let  $Y \subset {}^{red}(\mathcal{X}_{cl})$  be a closed embedding. (Note that such a  $Y$  is automatically locally of finite type.)

By Proposition 8.1.8, in order to prove Theorem 9.1.2, it suffices to show that the formal completion  $\mathcal{X}_Y^\wedge$  is formally smooth. Moreover, by Proposition 8.2.2 and Lemma 8.3.3, we can assume that  $Y$  is affine.

We will prove that  $\mathcal{X}_Y^\wedge$  is formally smooth by quoting/reproving the following result of [BD], Proposition 7.12.22. This proposition will also be useful to us in the sequel.

**Proposition 9.5.2.** *Let  $\mathcal{Z}_{cl}$  be a classical formal scheme, which as a classical indscheme is locally of finite type and  $\aleph_0$ . Assume that  $\mathcal{Z}_{cl}$  is classically formally smooth. Assume also that  ${}^{red}(\mathcal{Z}_{cl})$  is affine.*

*Then  $\mathcal{Z}_{cl}$  is isomorphic to retract of a filtered colimit (taken in  ${}^{cl}\text{Stk}$ ) of classical formal schemes each of which is isomorphic to the classical formal scheme underlying a formal DG scheme of Section 8.4.4.*

Let us deduce Theorem 9.1.2 from Proposition 9.5.2. This will be done via a series of lemmas:

*Proof.* (of Theorem 9.1.2)

**Lemma 9.5.3.** *For any  $\mathcal{X}_{cl} \in {}^{\leq 0}\text{DGindSch}_{\text{lft}}$  and  $\mathcal{X} := {}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(\mathcal{X}_{cl})$ , a reduced classical scheme  $Y$  and a closed embedding  $Y \subset {}^{red}(\mathcal{X}_{cl})$ , the canonical map*

$${}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}({}^{cl}(\mathcal{X}_Y^\wedge)) \rightarrow \mathcal{X}_Y^\wedge$$

*is an isomorphism.*

*Proof.* Let  ${}^{cl}\mathcal{X} \simeq \text{colim}_\alpha X_{\alpha, cl}$ . Let  $X_\alpha := {}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}(X_{\alpha, cl})$ , so

$$\mathcal{X} \simeq \text{colim}_\alpha X_\alpha,$$

where the colimit is taken in  $\text{Stk}$ .

Without loss of generality, we can assume that  $Y$  is contained in each  $X_{\alpha, cl}$ . Then

$${}^{cl}(\mathcal{X}_Y^\wedge) \simeq \text{colim}_\alpha {}^{cl}((X_\alpha)_Y^\wedge),$$

and the left-hand side in the lemma is

$$\text{colim}_\alpha {}^L\text{LKE}_{\text{Sch}^{\text{aff}} \hookrightarrow \text{DGSch}^{\text{aff}}}({}^{cl}((X_\alpha)_Y^\wedge)),$$

where the colimit is taken in  $\text{Stk}$ .



We now claim that the map

$$L\left(\operatorname{colim}_{\alpha}((X_{\alpha})_Y^{\wedge})\right) \rightarrow \left(L(\operatorname{colim}_{\alpha} X_{\alpha})\right)_Y^{\wedge} = \mathcal{X}_Y^{\wedge},$$

where both colimits are taken in  $\operatorname{PreStk}$ , is an isomorphism. This follows from Lemma 6.2.4 and Sect. 6.1.3(iii), combined with the fact that

$${}^{cl,red}(\operatorname{colim}_{\alpha} X_{\alpha}) \rightarrow {}^{cl,red}\left(L(\operatorname{colim}_{\alpha} X_{\alpha})\right)$$

is an isomorphism (where the colimits are again taken in  $\operatorname{PreStk}$ ). The latter follows from Lemma 1.1.5 and [GL:Stacks], Corollary 2.4.6.

Thus, the right-hand side in the lemma identifies with  $\operatorname{colim}_{\alpha} (X_{\alpha})_Y^{\wedge}$ , where the colimit is taken in  $\operatorname{Stk}$ .

Hence, to prove the lemma, it suffices to show that for every  $\alpha$ , the canonical map

$${}^L\operatorname{LKE}_{\operatorname{Sch}^{\text{aff}} \hookrightarrow \operatorname{DGSch}^{\text{aff}}} {}^{cl}((X_{\alpha})_Y^{\wedge}) \rightarrow (X_{\alpha})_Y^{\wedge}$$

is an isomorphism. However, this is the content of Proposition 6.7.2, which is applicable since  $X_{\alpha,cl}$  is of finite type, and in particular, Noetherian.  $\square$

**Lemma 9.5.4.** *Let  $\mathcal{Z}_{cl}$  be as in Proposition 9.5.2. Then*

$${}^L\operatorname{LKE}_{\operatorname{Sch}^{\text{aff}} \hookrightarrow \operatorname{DGSch}^{\text{aff}}}(\mathcal{Z}_{cl})$$

*is a formally smooth DG indscheme.*

Let us assume this lemma and finish the proof of Theorem 9.1.2. Indeed, it is enough to show that for  $\mathcal{X}_{cl}$  as in the theorem, the left-hand side in Lemma 9.5.3 is formally smooth. However, this follows from Lemma 9.5.4 for  $\mathcal{Z}_{cl} := {}^{cl}(\mathcal{X}_Y^{\wedge})$ .  $\square$

**9.5.5. Proof of Lemma 9.5.4.** Since the notion of formal smoothness is stable under taking retracts, we can assume that

$$\mathcal{Z}_{cl} \simeq \operatorname{colim}_{\alpha} \mathcal{Z}_{\alpha,cl},$$

(colimit taken in  ${}^{cl}\operatorname{Stk}$ ), where each  $\mathcal{Z}_{\alpha,cl}$  is isomorphic to the classical formal scheme underlying an affine formal DG scheme of Section 8.4.4.

Hence,  $\mathcal{Z} := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\text{aff}} \hookrightarrow \operatorname{DGSch}^{\text{aff}}}(\mathcal{Z}_{cl})$  is isomorphic to

$$\operatorname{colim}_{\alpha} \mathcal{Z}_{\alpha},$$

where the colimit is taken in  $\operatorname{Stk}$ , where

$$\mathcal{Z}_{\alpha} := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\text{aff}} \hookrightarrow \operatorname{DGSch}^{\text{aff}}}(\mathcal{Z}_{\alpha,cl}).$$

By Proposition 8.4.6 combined with Corollary 8.4.5, each  $\mathcal{Z}_{\alpha}$  is a formally smooth DG indscheme. Thus, it remains to prove the following:

**Lemma 9.5.6.** *Let  $\alpha \mapsto \mathcal{Z}_{\alpha}$  be filtered family of objects of  $\operatorname{Stk}$ , each of which is formally smooth as an object of  $\operatorname{PreStk}$ . Assume that for every  $n$ , all  $\mathcal{Z}_{\alpha|_{\leq n \operatorname{DGSch}^{\text{aff}}}}$  are  $k$ -truncated for some  $k$ . Then  $\mathcal{Z} := \operatorname{colim}_{\alpha} \mathcal{Z}_{\alpha}$  is also formally smooth as an object of  $\operatorname{PreStk}$ , where the colimit is taken in  $\operatorname{Stk}$ .*

*Proof.* Consider the object  $\mathcal{Z}' := \operatorname{colim}_{\alpha} \mathcal{Z}_{\alpha}$ , where the colimit is taken in  $\operatorname{PreStk}$ . Since homotopy groups commute with filtered colimits, we obtain that  $\mathcal{Z}'$  is formally smooth as an object of  $\operatorname{PreStk}$ . In particular, it is convergent. It remains to show that the canonical map  $\mathcal{Z}' \rightarrow \mathcal{Z}$  is an isomorphism. To show this, it suffices to show that  $\mathcal{Z}'$  satisfies descent. By convergence, it is enough to check the descent condition on  ${}^{\leq n}\operatorname{DGSch}^{\operatorname{aff}}$ . But the latter follows from the truncatedness assumption by Lemma 1.1.5.  $\square$

$\square$ (Lemma 9.5.4)

**9.6. Proof of the key proposition.** In this subsection we will prove Proposition 9.5.2, reproducing a slightly modified argument from [BD], pages 328-331. <sup>24</sup>

*Remark.* Note that the statement of Proposition 7.1.22 of [BD] is slightly stronger: it asserts that, under the (innocuous) additional assumption that  ${}^{\operatorname{red}}(\mathcal{Z}_{cl})$  is connected, we have an isomorphism

$$\mathcal{Z}_{cl} \simeq \mathcal{Z}_{0,cl} \times {}^{cl}(\operatorname{Spf}(k[[z_1, z_2, \dots]])),$$

where  $\mathcal{Z}_{0,cl}$  is isomorphic to the classical formal scheme underlying a formal DG scheme of Section 8.4.4. The reason we choose the formulation given in Proposition 9.5.2 is that it makes it more amenable for generalization in the non-finite type situation.

**9.6.1. Step 0: initial remarks.** Denote  $Z := {}^{\operatorname{red}}(\mathcal{Z}_{cl})$  and  $\mathcal{Z} := {}^L\operatorname{LKE}_{\operatorname{Sch}^{\operatorname{aff}} \hookrightarrow \operatorname{DGSch}^{\operatorname{aff}}}(\mathcal{Z}_{cl})$ .

By Proposition 5.3.2, the finite type condition implies that the object  $T^*\mathcal{Z}|_Z$  belongs to  $\operatorname{Pro}(\operatorname{Coh}(Z)^{\leq 0})$ , where

$$T^*\mathcal{Z}|_Z := T_z^*\mathcal{Z},$$

where  $z : Z \rightarrow \mathcal{Z}$  is the tautological map. Proposition 5.2.3 implies that  $T^*\mathcal{Z}|_Z$  is  $\aleph_0$  as an object of  $\operatorname{Pro}(\operatorname{Coh}(Z)^{\leq 0})$ .

By Proposition 9.4.2 and Proposition 7.12.6(iii) of [BD],  $T^*\mathcal{Z}|_Z$  is the dual of a Mittag-Leffler quasi-coherent sheaf  $\mathcal{M}$  on  $Z$ . By Theorem 7.12.8 of [BD], the  $\aleph_0$  condition implies that  $\mathcal{M}$  is actually projective.

Multiplying  $\mathcal{Z}$  by a suitable formally smooth classical indscheme as in [BD], Proposition 7.12.14, we can assume that  $\mathcal{M}$  is a free countably generated  $\mathcal{O}_Z$ -module. <sup>25</sup> Thus, we obtain that

$$(9.4) \quad H^0(T^*\mathcal{Z}|_Z) \in \operatorname{Pro}(\operatorname{Coh}(Z)^{\heartsuit})$$

can be represented as  $P := \varinjlim_{k \in \mathbb{N}} P_k$ , where  $P_k$  are locally free (in fact, free) sheaves on  $Z$  of finite rank, and the maps  $P_{k+1} \rightarrow P_k$  are surjective.

Let us write  $\mathcal{Z}_{cl} \simeq \operatorname{colim}_{n \in \mathbb{N}} Z_n$  with  $Z = Z_0$ . Let  $\mathcal{I}_n$  denote the sheaf of ideals of  $Z$  in  $Z_n$ . The finite type hypothesis implies that  $\mathcal{I}_n \in \operatorname{Coh}(Z_n)^{\heartsuit}$ . Consider  $\mathcal{I}_n|_Z \simeq \mathcal{I}_n/\mathcal{I}_n^2 \in \operatorname{Coh}(Z)^{\heartsuit}$  and denote

$$\mathcal{J}|_Z := \varinjlim_n \mathcal{I}_n|_Z \in \operatorname{Pro}(\operatorname{Coh}(Z)^{\heartsuit}).$$

By Proposition 9.4.2, the long exact cohomology sequence for the map  $Z \hookrightarrow \mathcal{Z}$  gives rise to a 4-term exact sequence in  $\operatorname{Pro}(\operatorname{Coh}(Z)^{\heartsuit})$ :

$$(9.5) \quad 0 \rightarrow H^{-1}(T^*Z) \rightarrow \mathcal{J}|_Z \rightarrow H^0(T^*\mathcal{Z}|_Z) \rightarrow H^0(T^*Z) \rightarrow 0.$$

<sup>24</sup>The reason that we include the proof instead of just quoting the result from [BD] is that it seems that the considerations that involve derived cotangent spaces that we introduce help to make the argument of *loc. cit.* more conceptual.

<sup>25</sup>This last procedure is the reason the word "retract" appears in the formulation of Proposition 9.5.2.

9.6.2. *Step 1: "the finite-dimensional case".* Let us first assume that  $H^0(T^*\mathcal{Z}|_Z)$  is an object of  $\text{Coh}(Z)$ . In this case we will prove that  ${}^{cl}\mathcal{Z}$  is isomorphic to the classical indscheme underlying a DG indscheme of the example in Section 8.4.4.

By (9.4),  $H^0(T^*\mathcal{Z}|_Z)$  is locally free of finite rank over  $Z$ .

As in [BD], top of page 329, it suffices to show that the system of coherent sheaves  $n \mapsto \mathcal{J}_n|_Z$  stabilizes. However, (9.5) implies that  $\mathcal{J}|_Z$  is in fact an object of  $\text{Coh}(Z)^\heartsuit$ . Since the maps  $\mathcal{J}_{n+1}|_Z \rightarrow \mathcal{J}_n|_Z$  are surjective, this implies the stabilization statement.

9.6.3. *Step 2: choosing generators for the ideal.* For a general  $\mathcal{Z}_{cl}$  we will construct an object  $Q \in \text{Pro}(\text{Coh}(Z)^\heartsuit)$  of the form

$$Q \simeq \varprojlim_{m \in \mathbb{N}} Q_m,$$

where  $Q_m$  are locally free sheaves on  $Z$  of finite rank, and the maps  $Q_{m+1} \rightarrow Q_m$  are surjective, and a map  $f : Q \rightarrow \mathcal{J}|_Z$ , such that the composition

$$Q \rightarrow \mathcal{J}|_Z \rightarrow H^0(T^*\mathcal{Z}|_Z) =: P$$

is injective and has the property that  $\text{coker}(Q \rightarrow P)$  belongs to  $\text{Coh}(Z)$  and is locally free.

Consider again (9.5). Let  $k$  be an index such that the map  $P \rightarrow H^0(T^*Z)$  factors through a map  $P_k \rightarrow H^0(T^*Z)$ , and let  $Q := \ker(P \rightarrow P_k)$ . Set

$$R := \mathcal{J}|_Z \times_P Q.$$

By construction, the map  $R \rightarrow Q$  is surjective, i.e., we have the following short exact sequence in  $\text{Pro}(\text{Coh}(Z)^\heartsuit)$ :

$$0 \rightarrow H^1(T^*Z) \rightarrow R \rightarrow Q \rightarrow 0.$$

Since  $Q$  is pro-projective and the category of indices is  $\mathbb{N}$ , the map  $R \rightarrow Q$  admits a right inverse, which gives rise to the desired map  $f : Q \rightarrow R \rightarrow \mathcal{J}|_Z$ .

9.6.4. *Step 3.* We shall now use the above pair  $(Q, f : Q \rightarrow \mathcal{J}|_Z)$  to construct the desired family of sub-indchemes of  $\mathcal{Z}_{cl}$ , each being as in Step 1.

For every  $n$  consider the object

$$\mathcal{J}|_{Z_n} := \varprojlim_{n' \geq n} \mathcal{J}_{n'}|_{Z_n} \in \text{Pro}(\text{Coh}(Z_n)^\heartsuit).$$

We can extend the locally free sheaves  $Q_m$  to a compatible family of locally free finite rank coherent sheaves  $n \mapsto Q_m|_{Z_n}$  and a compatible family of maps

$$f|_{Z_n} : Q|_{Z_n} := \varprojlim_m Q_m|_{Z_n} \rightarrow \mathcal{J}|_{Z_n}.$$

Let  $Q^m|_{Z_n}$  be the kernel of the map  $Q|_{Z_n} \rightarrow Q_m|_{Z_n}$ . For each  $m$  we define the closed sub-scheme  $Z_n^m$  of  $Z_n$  to be given by the ideal  $\mathcal{J}_n^m$  equal to the image of

$$Q^m|_{Z_n} \hookrightarrow Q|_{Z_n} \xrightarrow{f|_{Z_n}} \mathcal{J}|_{Z_n} \rightarrow \mathcal{J}_n.$$

We set

$$\mathcal{Z}_{cl}^m := \text{colim}_n Z_n^m.$$

It is clear from the construction that

$$\mathcal{Z}_{cl} \simeq \text{colim}_m \mathcal{Z}_{cl}^m.$$

9.6.5. *Step 4.* It remains to show that each  $\mathcal{Z}_{cl}^m$  is a classical indscheme satisfying the assumptions of Step 1. Let

$$\mathcal{Z}^m := {}^L\mathrm{LKE}_{\mathrm{Sch}^{\mathrm{aff}} \hookrightarrow \mathrm{DGSch}^{\mathrm{aff}}}(\mathcal{Z}_{cl}^m).$$

Using Corollary 9.4.5, it suffices to show that  $H^0(T^*\mathcal{Z}^m|_Z) \in \mathrm{Pro}(\mathrm{Coh}(Z)^\heartsuit)$  is locally free of finite rank and that  $H^{-1}(T^*\mathcal{Z}^m|_Z) = 0$ .

For that it suffices to show that the map

$${}^{\text{“}lim\text{”}}\mathcal{J}_n^m|_Z \rightarrow H^0(T^*\mathcal{Z}|_Z)$$

is injective and that the quotient belongs to  $\mathrm{Coh}(Z)^\heartsuit$  and is locally free of finite rank. However, by construction, we have a surjective map

$$Q^m := \ker(Q \rightarrow Q_m) \twoheadrightarrow {}^{\text{“}lim\text{”}}\mathcal{J}_n^m|_Z,$$

and the required properties follow from the corresponding properties of the map  $f$ .

□(Proposition 9.5.2)

## 10. QCoh AND IndCoh ON FORMALLY SMOOTH INDSCHEMES

The goal of this section is to prove the following result, originally established by J. Lurie using a different method:

**Theorem 10.0.6.** *Let  $\mathcal{X}$  be a formally smooth DG indscheme, which is weakly  $\aleph_0$  and locally almost of finite type. Then the functor*

$$\Psi_{\mathcal{X}}^\vee := - \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}} : \mathrm{QCoh}(\mathcal{X}) \rightarrow \mathrm{IndCoh}(X)$$

*is an equivalence.*

Note that by Theorem 9.1.2, the DG indscheme  $\mathcal{X}$  is 0-coconnected, so  $\mathrm{QCoh}(\mathcal{X})$  is canonically equivalent to  $\mathrm{QCoh}({}^L\tau^{cl}(\mathcal{X}))$ , which itself is equivalent to  $\mathrm{QCoh}(\tau^{cl}(\mathcal{X}))$  by [GL:QCoh], Corollary 1.3.7.

Also note that by Corollary 2.4.2, we obtain that for  $\mathcal{X}$  as in the theorem, the category  $\mathrm{QCoh}(\mathcal{X})$  is compactly generated.

The rest of this section is devoted to the proof of Theorem 10.0.6.

### 10.1. Reduction to the “standard” case.

10.1.1. Write  ${}^{cl}\mathcal{X}$  as  $\mathrm{colim}_{\alpha} X_{\alpha}$ , where  $X_{\alpha}$  are classical schemes locally of finite type. Let  $\mathcal{X}_{\alpha} := \mathcal{X}_{red X_{\alpha}}^{\wedge}$  be the formal completion of  $\mathcal{X}$  along  ${}^{red}X_{\alpha}$ . Each  $\mathcal{X}_{\alpha}$  is DG indscheme satisfying the assumptions of the theorem.

Since

$$\mathrm{colim}_{\alpha} \mathcal{X}_{\alpha} \rightarrow \mathcal{X}$$

is an isomorphism, the functors

$$\mathrm{QCoh}(\mathcal{X}) \rightarrow \lim_{\alpha} \mathrm{QCoh}(\mathcal{X}_{\alpha}) \text{ and } \mathrm{IndCoh}(\mathcal{X}) \rightarrow \lim_{\alpha} \mathrm{IndCoh}(\mathcal{X}_{\alpha})$$

are both equivalences, where the first limit is taken with respect to the  $*$ -pullback functors, and the second limit is taken with respect to the  $!$ -pullback functors.

Since for  $\alpha \rightarrow \beta$  the diagrams

$$\begin{array}{ccccc} \mathrm{QCoh}(\mathcal{X}) & \longrightarrow & \mathrm{QCoh}(\mathcal{X}_\beta) & \longrightarrow & \mathrm{QCoh}(\mathcal{X}_\alpha) \\ \downarrow - \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}} & & \downarrow - \otimes_{\mathcal{O}_{\mathcal{X}_\beta}} \omega_{\mathcal{X}_\beta} & & \downarrow - \otimes_{\mathcal{O}_{\mathcal{X}_\alpha}} \omega_{\mathcal{X}_\alpha} \\ \mathrm{IndCoh}(\mathcal{X}) & \longrightarrow & \mathrm{IndCoh}(\mathcal{X}_\beta) & \longrightarrow & \mathrm{IndCoh}(\mathcal{X}_\alpha) \end{array}$$

are commutative, it suffices to show that each of the functors

$$(10.1) \quad - \otimes_{\mathcal{O}_{\mathcal{X}_\alpha}} \omega_{\mathcal{X}_\alpha} : \mathrm{QCoh}(\mathcal{X}_\alpha) \rightarrow \mathrm{IndCoh}(\mathcal{X}_\alpha)$$

is an equivalence.

So, from now on we will assume that  $\mathcal{X}$  is formal.

10.1.2. By [GL:IndCoh], Sect. 9.4, the functor  $\mathrm{IndCoh}$  satisfies fpppf descent, and in particular, Zariski descent. So, the statement about equivalence in (10.1) is local in the Zariski topology. Therefore, we can assume that  $\mathcal{X}$  is affine, and thus apply Proposition 9.5.2.

10.1.3. Since the statement of the theorem survives taking retracts and colimits of DG ind-schemes, we can assume that  $\mathcal{X}$  is a formal DG scheme described in Sect. 8.4.4. The proof that the functor  $- \otimes_{\mathcal{O}_{\mathcal{X}}} \omega_{\mathcal{X}}$  is an equivalence in this case is a rather straightforward but somewhat tedious verification, which we shall presently perform.

## 10.2. The functor $\Psi_{\mathcal{X}}$ .

10.2.1. Recall the notation of Sect. 8.4.4. Let us denote by  $\mathcal{Y}$  the DG indscheme  $\mathbb{A}^{n,m}$ . Let  $f : \mathcal{X} \hookrightarrow \mathcal{Y}$  denote the corresponding closed embedding.

Since

$$\mathcal{X} \simeq 0 \times_{\mathbb{A}^k} \mathcal{Y},$$

by [GL:QCoh], Proposition 3.2.1, we have:

$$(10.2) \quad \mathrm{QCoh}(\mathcal{X}) \simeq \mathrm{Vect} \bigotimes_{\mathrm{QCoh}(\mathbb{A}^k)} \mathrm{QCoh}(\mathcal{Y}).$$

By Sect. 7.2, the indscheme  $\mathcal{Y}$  is quasi-perfect (i.e., the category  $\mathrm{QCoh}(\mathcal{Y})$  is compactly generated and its compact objects are perfect). We claim:

**Corollary 10.2.2.** *The DG indscheme  $\mathcal{X}$  is quasi-perfect.*

*Proof.* From (10.2) we obtain that a generating set of compact objects of  $\mathrm{QCoh}(\mathcal{X})$  is obtained as the essential image under the functor  $f^*$  of compact objects of  $\mathrm{QCoh}(\mathcal{Y})$ . The assertion of the lemma follows from the fact that the pullback functor preserves perfectness.  $\square$

In particular, the functor

$$\Psi_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{QCoh}(\mathcal{X})$$

is well-defined. Moreover, showing that  $\Psi_{\mathcal{X}}^{\vee}$  is an equivalence is equivalent to showing that  $\Psi_{\mathcal{X}}$  is an equivalence.

10.2.3. Let  $\sim \mathcal{Y} := \sim \mathbb{A}^{n,m}$  be the *scheme* introduced in the course of the proof of Proposition 8.4.6. Let  $\sim \mathcal{X}$  be the DG *scheme*

$$0 \times_{\mathbb{A}^k} \sim \mathcal{Y}.$$

As we saw in Sect. 8.4.4, the DG scheme  $\mathcal{X}$  is also 0-coconnective.

The formal (DG) scheme  $\mathcal{X}$  is obtained as a formal completion of  $\sim \mathcal{X}$  along a Zariski-closed subset  $X$ . Let  $\hat{i} : \mathcal{X} \rightarrow \sim \mathcal{X}$  denote the corresponding map, and let

$$\sim \mathcal{X} - X =: U_{\mathcal{X}} \xrightarrow{j} \sim \mathcal{X}$$

be the complementary open embedding.

Since the DG scheme  $\sim \mathcal{X}$  is Noetherian, the category  $\mathrm{IndCoh}(\sim \mathcal{X})$  and the functor

$$\Psi_{\mathcal{X}} : \mathrm{IndCoh}(\mathcal{X}) \rightarrow \mathrm{QCoh}(\mathcal{X})$$

are well-defined (see [GL:IndCoh], Proposition 1.4.3).

10.2.4. We will deduce the fact that  $\Psi_{\mathcal{X}}$  is an equivalence from the following statement:

**Proposition 10.2.5.** *The diagram of functors*

$$\begin{array}{ccc} \mathrm{IndCoh}(\mathcal{X}) & \xrightarrow{\hat{i}_*^{\mathrm{IndCoh}}} & \mathrm{IndCoh}(\sim \mathcal{X}) \\ \Psi_{\mathcal{X}} \downarrow & & \Psi_{\sim \mathcal{X}} \downarrow \\ \mathrm{QCoh}(\mathcal{X}) & \xrightarrow{\hat{i}^?} & \mathrm{QCoh}(\sim \mathcal{X}) \end{array}$$

*commutes.*

*Remark.* This proposition does not formally follow from the commutativity of (7.11), because the latter relied on the finite type assumption of the ambient DG scheme (in our case the ambient scheme is  $\sim \mathcal{X}$ , and it is not of finite type).

Let us assume this proposition for a moment and finish the proof of the fact that  $\Psi_{\mathcal{X}}$  is an equivalence (and thereby of Theorem 10.0.6).

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(\sim \mathcal{X}) & \xrightarrow{j^{\mathrm{IndCoh},*}} & \mathrm{IndCoh}(U_{\mathcal{X}}) \\ \Psi_{\sim \mathcal{X}} \downarrow & & \downarrow \Psi_{U_{\mathcal{X}}} \\ \mathrm{QCoh}(\sim \mathcal{X}) & \xrightarrow{j^*} & \mathrm{QCoh}(U_{\mathcal{X}}) \end{array}$$

(see [GL:IndCoh], Proposition 3.3.4).

By Proposition 7.1.3, the category  $\mathrm{QCoh}(\mathcal{X})$  identifies with the kernel of the functor

$$j^* : \mathrm{QCoh}(\sim \mathcal{X}) \rightarrow \mathrm{QCoh}(U_{\mathcal{X}}).$$

By Proposition 7.4.5, the category  $\mathrm{IndCoh}(\mathcal{X})$  identifies with the kernel of the functor

$$j^{\mathrm{IndCoh},*} : \mathrm{IndCoh}(\sim \mathcal{X}) \rightarrow \mathrm{IndCoh}(U_{\mathcal{X}}).$$

(We remark that in Proposition 7.4.5 it was assumed that the ambient scheme is almost of finite type over the field, but the proof applies in the case when it is only assumed Noetherian, which is the case for  $\sim \mathcal{X}$ .)

The required assertion follows from the fact that the functors  $\Psi_{\sim \mathcal{X}}$  and  $\Psi_{U_{\mathcal{X}}}$  are equivalences, since the corresponding DG schemes are 0-coconnective and the underlying classical schemes are regular (see [GL:IndCoh], Lemma 1.1.6).  $\square$

**10.3. Proof of Proposition 10.2.5.** We shall compare the the functors

$$\mathrm{QCoh}(\sim \mathcal{X}) \otimes \mathrm{IndCoh}(\mathcal{X}) \rightrightarrows \mathrm{Vect}$$

that arise from the two circuits of the diagram and the duality pairing

$$\langle -, - \rangle_{\mathrm{QCoh}(\sim \mathcal{X})} : \mathrm{QCoh}(\sim \mathcal{X}) \otimes \mathrm{QCoh}(\sim \mathcal{X}) \rightarrow \mathrm{Vect}.$$

For  $\mathcal{F} \in \mathrm{QCoh}(\sim \mathcal{X})$  and  $\mathcal{F}' \in \mathrm{IndCoh}(\mathcal{X})$  we have:

$$\langle \mathcal{F}, \hat{i}_? \circ \Psi_{\mathcal{X}}(\mathcal{F}_1) \rangle_{\mathrm{QCoh}(\sim \mathcal{X})} \simeq \langle \hat{i}^*(\mathcal{F}), \Psi_{\mathcal{X}}(\mathcal{F}_1) \rangle_{\mathrm{QCoh}(\mathcal{X})} \simeq \Gamma^{\mathrm{IndCoh}}(\mathcal{X}, \hat{i}^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}_1),$$

where the first isomorphism follows from Corollary 7.2.6, and the second one from Sect. 7.5.2.

However, the description of the functor  $\hat{i}_*^{\mathrm{IndCoh}}$  given in Sect. 7.4.3 (which is valid for all Noetherian schemes) implies that we have a canonical isomorphism

$$\Gamma^{\mathrm{IndCoh}}(\mathcal{X}, -) \simeq \Gamma^{\mathrm{IndCoh}}(\sim \mathcal{X}, -) \circ \hat{i}_*^{\mathrm{IndCoh}}.$$

Hence, the above expression can be further rewritten as

$$\Gamma^{\mathrm{IndCoh}} \left( \sim \mathcal{X}, \hat{i}_*^{\mathrm{IndCoh}}(\hat{i}^*(\mathcal{F}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{F}_1) \right),$$

which by the projection formula is canonically isomorphic to

$$\Gamma^{\mathrm{IndCoh}} \left( \sim \mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_{\sim \mathcal{X}}} \hat{i}_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \right).$$

Now,

$$\langle \mathcal{F}, \Psi_{\sim \mathcal{X}} \circ \hat{i}_*^{\mathrm{IndCoh}}(\mathcal{F}_1) \rangle_{\mathrm{QCoh}(\sim \mathcal{X})} \simeq \Gamma^{\mathrm{IndCoh}}(\sim \mathcal{X}, \mathcal{F} \otimes_{\mathcal{O}_{\sim \mathcal{X}}} \hat{i}_*^{\mathrm{IndCoh}}(\mathcal{F}_1)),$$

as required.  $\square$

## REFERENCES

- [BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, available at <http://math.uchicago.edu/~mitya/langlands.html>.
- [Dr] V. Drinfeld, *Infinite-dimensional vector bundles*, in: Algebraic Geometry and Number Theory, Progr. Math. **253** (2006), 264–304, Birkhauser Boston.
- [EF] D. Eisenbud and E. Frenkel, appendix to the paper: M. Mustata, *Jet schemes of locally complete intersection canonical singularities*, Invent. Math. **145**, 397–424 (2001).
- [GL:DG] D. Gaitsgory, *Notes on Geometric Langlands: Generalities on DG categories*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GL:Stacks] D. Gaitsgory, *Notes on Geometric Langlands: Stacks*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GL:QCoh] D. Gaitsgory, *Notes on Geometric Langlands: Quasi-coherent sheaves on stacks*, available at <http://www.math.harvard.edu/~gaitsgde/GL/>.
- [GL:IndCoh] D. Gaitsgory, *Notes on Geometric Langlands: Ind-coherent sheaves*, arXiv:1105.4857.
- [GL:Funct] D. Gaitsgory and N. Rozenblyum, *Notes on Geometric Langlands: Further functoriality for Ind-coherent sheaves*, in preparation.
- [Lu0] J. Lurie, *Higher Topos Theory*, Princeton Univ. Press (2009).
- [Lu1] J. Lurie, *DAG-VIII*, available at <http://www.math.harvard.edu/~lurie>.
- [RG] M. Raynaud and L. Gruson, *Critères de platitude et de projectivité*, Invent. Math., **13**, 1–89 (1971).